Solutions to Assignment 4

1. Following the hint from class, we examine all the terms in one free variable you can define up to equivalence. These are all the terms one can form with a variable x using the constant 0 and the functions S, $+$ and $\times$. This yields all polynomials with natural number coefficients.
   However, we know that the function 2^x is primitive recursive and 2^x grows faster than any polynomial as x tends ~~to~~ to infinity. So 2^x is not represented as a term.

2. Again, following the hint from class, suppose we are considering just one schema and we are confronted with a string of L_A symbols. The question is whether the given string is an instance of the schema. To be an instance of the schema, the string we are considering must be the substitution of specific formulas for the schema variables into the given schema. So if the string you are looking at <u>is</u> an instance of the schema then the necessary substitutions are substrings of the given string. A terrible but decidable way to proceed then is to assign arbitrary substrings to the variables, check if they are formulas, if so substitute them into the schema and see if you get the original string. If so, the string is an axiom. Since there are only finitely many ~~st~~ substrings and finitely many schema, this process is finite (albeit long!).

3. Using Gödel's β-function, we can define $y = 2^x$ by:

$$\exists c, d \, \big(\beta(c,d,0) = 1 \;\wedge$$
$$\forall z < x \, \big(\beta(c,d,S(z)) = 2 \times \beta(c,d,z) \big) \;\wedge$$
$$y = \beta(c,d,x) \;\big).$$

4. a) First we show that if $x + y = 0$ then $y = -x$.

$x + y = 0$ so $(-x) + x + y = -x$ (no need for brackets by associativity).

so $x + (-x) + y = -x$ (commutativity)

and so $y = -x$ (axioms i) and iv)).

So since $-(-x) + (-x) = 0$ and $x + (-x) = 0$ we have $x = -(-x)$. Similarly,

$(-x) + (-y) + x + y = 0$ so $-(x+y) = (-x) + (-y)$.

b) Using associativity and commutativity, we can group the variables together to get each term in the form:

$$k_1 x_1 + l_1(-x_1) + k_2 x_2 + l_2(-x_2) + \cdots + k_j x_j + l_j(-x_j).$$

Now if say $k_1 \geq l_1$ then by pairing l_1 x_1's with l_1 $(-x_1)$'s we get that $k_1 x_1 + l_1(-x_1) = (k_1 - l_1)x_1$ (if $k_1 = l_1$ then this is just 0). If $k_1 < l_1$ then we do the same and are left with $(k_1 - l_1)x_1$.

c) This follows immediately from b).

d). The only thing to show here is that the definition of $y = rx$ is well-defined. If $r = \frac{p}{q}$ then by the divisibility axiom for q, given any x, there is a unique y s.t. $px = qy$. The only ambiguity is, what if we also represent r as $\frac{p'}{q'}$; do we get the same y?

We can assume $p' = ps$ and $q' = qs$. So suppose $qy = px$ for some x and y. Then $sqy = spx$ and we have $q'y = p'x$ as well. So this function is well-defined.

e) We now show that every formula is equivalent to a quantifier free formula by induction on formulas:

i) Atomic formulas are quantifier-free.

ii) Closure under the connectives does not add quantifiers. Moreover, from the hint, we can assume the q.f.f. is in the form
(D.N.F.) $\bigvee_{i \in I} \bigwedge_{j \in J_i} \varphi_j$ where φ_j is either atomic or negated atomic.

iii) Since $\forall x\, \varphi$ is equivalent to $\neg \exists x \neg \varphi$, it suffices to show any formula of the form $\exists x\, \varphi$ where φ is q.f. is equivalent to one which is q.f.

As mentioned above, we may assume φ is in D.N.F. Since $\exists x \bigvee_{i \in I} \varphi_i$ is equivalent to $\bigvee_{i \in I} \exists x\, \varphi_i$, we can further assume that our formula has the form:

$\exists x \bigwedge_{j \in J} \varphi_j$ where each φ_j is either atomic or negated atomic.

There are now two cases: ① At least one of the φ_j's is an atomic formula. In this case, say the atomic formula looks like $nx + \sigma = 0$ where σ is a term in other variables. What we want to do is solve for x i.e. let $x = -\frac{1}{n}\sigma$ (which we know is uniquely determined by d_s). What we really do is replace any term

$$mx + \tau \quad \text{by} \quad -m\sigma + n\tau$$

The second term will be 0 iff the first is. In this way we eliminate all x's from the formulas φ_j. If we call the substituted formulas $\tilde{\varphi}_j$ then $\exists x \bigwedge \varphi_j$ is equivalent to $\bigwedge \tilde{\varphi}_j$ and so is quantifier free.

Thankfully, case ② is faster: If all φ_j are negated atomic formulas, this just says x is not equal to finitely many things. Since $\mathbb{Q}$ is infinite, this formula is just true.

We can say it is equivalent to $0 = 0$.

f) So finally, the theory of Q is complete and decidable. If we are given any sentence true in Q, we can eliminate all the quantifiers as above. But it is a sentence and so it is a quantifier-free formula with no free variables. Again, using D.N.F. we see we only care about atomic and negated atomic formulas. The only such sentences are $0 = 0$ and $0 \neq 0$. Using truth tables then we can determine which sentences are true and which are false. Since we have an explicit axiomatization, we have decidability.