Solutions to Assignment 3

We have the addition table for $\mathbb{N}^*$ in Smith given by:

$+^*$ behaves as $+$ on $\mathbb{N}$

$a +^* n = a$, $b +^* n = b$ for all $n \in \mathbb{N}$

$x +^* a = b$, $x +^* b = a$ for all $x \in \mathbb{N}^*$.

We want to define $\times^*$ so that axioms 6 and 7 hold in $\mathbb{N}^*$.

Let $\times^*$ agree with $\times$ on $\mathbb{N}$. To make axiom 6 hold we must have $a \times^* 0 = b \times^* 0 = 0$. For any ~~other~~ $n \in \mathbb{N}$ with $n \neq 0$, $n = S(m)$ and so we will want

$a \times^* S(m) = a \times^* m +^* a = b$ so let $a \times^* n = b$

Similarly, let $b \times^* n = a$.

In $\mathbb{N}^*$, $S^*(a) = a$ and $S^*(b) = b$ so we will want

$x \times^* S^*(a) = x \times^* a +^* x$ for all $x \in \mathbb{N}^*$ so when $n \in \mathbb{N}$, let $n \times^* a = a$, with also $a \times^* a = b$ and $b \times^* a =$ b.

We also want $x \times^* S^*(b) = x \times^* b +^* x$ so let $n \times^* b = b$ for all $n \in \mathbb{N}$, $a \times^* b = b$ and $b \times^* b = a$.

2. We show that the negation of a Π_1-formula is equivalent to a Σ_1-formula by induction on the formation of the Π_1-formula.

i) If φ is a Δ_0-formula then so is $\neg\varphi$ and so $\neg\varphi$ is also a Σ_1-formula.

ii) Suppose $\varphi = \varphi_1 \wedge \varphi_2$ where φ_1, φ_2 are Π_1-formulas. We ~~know~~ assume by induction that $\neg\varphi_1$ and $\neg\varphi_2$ are equivalent to Σ_1 formulas.

But $\neg\varphi$ is equivalent to $\neg\varphi_1 \vee \neg\varphi_2$ which by assumption is equivalent to a Σ_1-formula.

"$\vee$" is similar.

iii) Suppose $\varphi = \forall x \psi$ and assume $\neg\psi$ is equivalent to a Σ_1-formula. Then $\neg\varphi$ is equivalent to $\exists x \neg\psi$ which is equivalent to a Σ_1-formula by assumption.

3. The idea is that if we want to capture exponentiation we need a new function symbol $exp(x,y)$ intended to capture x^y. We also know that we can capture exp in a p.r. fashion from Q so the axioms should be those that allow one to define exp inductively:

So the axioms could be:

$$\exp(a, 0) = 1 \quad \text{and}$$

$$\exp(a, S(b)) = \exp(a, b) \times a.$$

One proves by induction that these axioms capture exp correctly on $\mathbb{N}$.

4. By reading the proof, one sees very little is being used except that T is consistent and at the very least knows that $0 \neq 1$. See Smith's comment (b) immediately above the Theorem. Assuming Baby Arithmetic is certainly enough as is assuming T contains Q.