Math 3TP3, Midterm Test

Bradd Hart, Feb. 16, 2024

There are 5 questions and each question is worth 5 marks; the test will be graded out of 25. Do as many problems as you wish.

Please write complete answers to all of the questions in the test booklet provided. Partial credit may be given for your work. Unless otherwise noted, you need to justify your solutions in order to receive full credit. Please be sure to include your name and student number on all sheets of paper that you hand in.

1. In the language L_A, write formulas which will express the following properties of the natural numbers; don't worry about exact syntax as long as your formula is readable.

   (a) x is prime.

   $$x \neq 0 \wedge x \neq 1 \wedge \forall y, z(y \times z = x \rightarrow y = x \vee z = x)$$

   (b) x is divisible by 3.

   $$\exists y(3 \times y = x)$$

   (c) $x \leq y \leq z$ i.e. a formula in three variables that expresses this relationship.

   $$\exists u, v(x + u = y \wedge y + v = z)$$

2. Determine if the following properties of n are decidable:

   (a) n is prime.

   This is decidable: check if there is any number greater than 1 and less than n which divides n.

   (b) The n^{th} decimal digit of e is 6.

   This is decidable. One can use the Taylor series to accurately approximate e to within 10^{-n} and determine if the n^{th} digit is 6.

   (c) n is even, greater than 2 and the sum of two primes. Bonus: What impact, if any, does knowing this have on the decidability of Goldbach's conjecture?

   This is also decidable. If n is even and greater than 2, check all primes less than n to see if two of them add up to n. Bonus: This says nothing about the decidability of the Goldbach conjecture. At best it tells us that the n which we can write as the sum of two primes is effectively enumerable.

3. Explain why it is possible to effectively enumerate all L_A-formulas.

   Every L_A-formula is a finite string of L_A symbols. It is possible to enumerate all finite strings of L_A symbols. One can certainly list all sequences of length 1, then length 2, etc. By dovetailing these lists together we can enumerate all finite strings. We now would just have to go through this list and for each finite string, determine if it is a formula, again something that can be done decidablly for each string.

4. Timeline: put the following mathematicians in order according to the date of their most significant mathematical achievement and write one sentence saying what that accomplishment was:

   Cantor, Euclid, Gödel, Hilbert, Leibniz, Weierstrass

   (a) Euclid - he wrote the Elements which was the standard mathematics textbook for 2,000 years.

   (b) Leibniz - he developed calculus at the same time as Newton; he thought about how to automate the calculations one had to perform routinely.

   (c) Weierstrass - gave a rigourous foundation for analysis including the δ-ε definition of limit

   (d) Cantor - developed set theory

   (e) Hilbert - gave an ICM address in 1900 asking for a proof of the consistency of arithmetic

   (f) Gödel - proved his incompleteness theorem in 1931 arguably showing that what Hilbert asked for was impossible.

5. Suppose that $g_1, g_2, g_3, \ldots$ is a list of computable functions on the natural numbers. Define the function f by:

$$f(n) = g_n(n) + 1.$$

   Show that f is not on the original list. What does this say about the original list and about the enumerability of the set of computable functions?

   Suppose that $f = g_n$ on the original list. Then $f(n) = g_n(n) + 1$ not $g_n(n)$ which is a contradiction. This says that the original list is in a complete list of the computable functions, and that there is no effectively computable list of computable functions.