Some final thoughts on Gödel's theorems

- Let's assume that we are talking about Peano arithmetic, PA, as our theory T.
- PA is consistent (or at least we think it is - didn't we prove that?) and it contains both Q and $I\Sigma_1$.
- So by the second incompleteness theorem it can't prove it's own consistency, $PA \nvdash Con_{PA}$.
- However, for every n,

$$PA \vdash \neg Prov_{PA}(n, \ulcorner 0 = 1 \urcorner).$$

- Why is that? $\neg Prov_{PA}(n, \ulcorner 0 = 1 \urcorner)$ is Σ_1 (think about it). So since it is true (n does not code a proof of 0=1), PA can prove it.
- But $\forall x \neg Prov_{PA}(x, \ulcorner 0 = 1 \urcorner)$ is just Con_{PA} so $PA \nvdash \forall x \neg Prov_{PA}(x, \ulcorner 0 = 1 \urcorner)$.

The issue

- What is going on here is that PA knows that any individual number n does not code a proof of 0 = 1 but the proof of this fact depends on n.
- When we want to check Con_{PA}, the universal quantifier involved is essentially asking for a uniform proof that is independent of the chosen n.
- For those of you that are looking at non-standard models of PA, this is one of those cases where it is not enough to check all $n \in N$ but we must check all “numbers” in all models of PA - standard ones and non-standard.
- The universal quantifier would then range over all standard and non-standard numbers.
- So if PA is consistent there would be some structure $N^\dagger$ in which PA holds and $\neg Con_{PA}$ also holds; that is, there is some “number” $n^* \in N^\dagger$ for which $Prov_{PA}(n^*, \ulcorner 0 = 1 \urcorner)$ holds. Very weird!

Hold it; can't you get around this by ...

- just starting with a theory T, constructing the Gödel sentence G_T and adding it as an axiom to T?
- You definitely know that G_T is true so this new theory would be consistent if T was a theory of arithmetic and the new theory would prove G_T; problem solved!
- No, not quite. The new theory, call it T' would have its own Gödel sentence $G_{T'}$ which would also be true but not provable from T'.
- You could try to do this inductively constructing more and more Gödel sentences for bigger and bigger theories but you would never catch your tail so to speak. Anything you do that can be captured in an effectively enumerable manner will not succeed - Gödel has already defeated this option.