Quick recap

- We have a theory of arithmetic T with the property that there is a Σ_1-formula $Prov_T(x, y)$ whose interpretation in N is $Prov_T(m, n)$ holds iff m is the code for a proof of the formula whose Gödel number is n.
- We have a diagonalization procedure which takes a formula $\varphi(x)$ and forms $\varphi(\ulcorner\varphi(x)\urcorner)$, the diagonalization of φ.
- It should also be clear that we can express the relation $Diag(x, y)$ whose interpretation is: x is the Gödel number of a formula $\varphi(x)$ and y is the Gödel number of the diagonalization of φ. If one thinks about how one would write a computer programme to determine if $Diag(m, n)$ holds, it should be clear that this can be written as a Σ_1-formula.

Recap cont'd

- After introducing the diagonalization, we introduced the relation $Gdl_T(x, y)$ where $Gdl_T(m, n)$ meant m is a code for a proof of the diagonalization of the formula coded by n. Given what we said on the first slide, it is easy to see that this is Σ_1:

$$Gdl_T(x, y) := \exists u \, Prov_T(x, u) \wedge Diag(y, u).$$

- Finally we formed the formula $\psi(y) := \forall x \, \neg Gdl_T(x, y)$ which is Π_1 and definitely not Σ_1.
- The Gödel sentence for T is then $G_T := \psi(\ulcorner \psi(x) \urcorner)$. This is a Π_1-sentence.
- We showed that if T is consistent then T does not prove G_T but G_T is true in N.

Various statements of the Main Theorem

Theorem (First version)

If T is a consistent theory of arithmetic which contains Q and for which there is a p.r relation for recognition of the axioms of T then there is a Π_1-sentence G_T which is true in N but which is not provable from T. In particular, T is not complete.

Theorem (Second version)

If T is an effectively enumerable theory of arithmetic which proves Q then there is a Π_1-sentence G_T true in N but which is not provable from T.

Main Theorem again

Theorem (Third version - most abstract)

There is no effectively enumerable, complete, consistent theory of arithmetic.

Compare this with what we had earlier in the term; recall that L_A is sufficiently expressive if for every effectively computable function f there is a formula $\varphi_f(x, y)$ such that $f(m) = n$ iff $\varphi(m, n)$ holds in N.

Theorem

If L_A is sufficiently expressive there there is no effectively enumerable, complete, consistent theory of arithmetic.