Gödel numbering

- Gödel devised a clever trick for coding the symbols in L_A as numbers.
- The key thing is to enumerate all the symbols in a decidable, one-to-one fashion starting from the number 1. For instance you could enumerate them

$$(,), \wedge, \vee, \neg, \rightarrow, \forall, \exists, +, x, S, 0, 1, =, x_0, x_1, x_2, \ldots$$

- Smith has his own numbering; the important thing is that to every symbol s we assign a unique number $\ulcorner s \urcorner > 0$ and this is done decidably.

Gödel numbering, cont'd

- If we have a string of symbols $s_0 s_1 s_2 \ldots s_n$ then we will code this string, its Gödel number $\ulcorner s_0, \ldots, s_n \urcorner$, as

$$2^{\ulcorner s_0 \urcorner} 3^{\ulcorner s_1 \urcorner} 5^{\ulcorner s_2 \urcorner} \ldots \pi(i)^{\ulcorner s_i \urcorner} \ldots \pi(n)^{\ulcorner s_n \urcorner}$$

  where $\pi(i)$ is the ith prime.
- There are several things to remember: π is a primitive recursive function. Moreover, the length of this number i.e. the number of primes involved in its factorization is $n + 1$ since the value of every symbol is greater than 0.
- We are able to tell the exponent of any prime in this factorization by using the p.r. function *exp*; recall that $exp(n, i)$ is the exponent of $\pi(i)$ in the prime factorization of n.

Gödel numbers of formulas and terms

- If we are given a term τ then it is just a string of symbols and so it has a Gödel number $\ulcorner \tau \urcorner$.
- For instance, if τ is $SS0$ then $\ulcorner SS0 \urcorner$ is

$$2^{\ulcorner S \urcorner} 3^{\ulcorner S \urcorner} 5^{\ulcorner 0 \urcorner}.$$

  Notice that this is a number at least as large as 180 even though it is coding the term for 2.
- Similarly for any formula φ, viewing it as a string of symbols, it has a Gödel number $\ulcorner \varphi \urcorner$.

Unravelling Gödel numbers

- We can do a number of things algorithmically and in a primitive recursive fashion.
- Given n, determine if n codes a string of symbols. This entails seeing if the prime factorization of n involves all the primes from 2 up to some given prime and then writing

$$n = 2^{k_0} 3^{k_1} \dots \pi(m)^{k_m}.$$

  We then decode $k_0, \dots, k_m$ as a string of symbols $s_0, \dots, s_m$.
- Once we have the symbols $s_0, \dots, s_m$ we have a decidable process for determining if this string is a formula or a term (or neither).