Assignment 4, Math 3TP3
Due Mar. 27, in class

1. We discussed the possibility of supplementing the language $L_A$ with new function symbols for all the primitive recursive functions. This would have saved us from having to define them with $\Sigma_1$-formulas. Show that although all primitive recursive functions can be expressed in $L_A$ they cannot all be expressed as terms in $L_A$.

2. A formula schema is defined inductively like a formula starting from formula variables which we will signify with capital letters $X_0, X_1, \ldots$. That is, formula schema are formed as follows:

   (a) $X_i$ is a formula schema.

   (b) Formula schema are closed under all propositional connectives: $\wedge, \vee, \neg$ and $\rightarrow$.

   (c) If $\Phi$ is a formula schema, so are $\forall x \Phi$ and $\exists x \Phi$.

   (d) All formula schema are formed through finite applications of the previous clauses.

   A theory is said to be described by finitely many axiom schema if the axioms are the result of substitution of formulas for formula variables in some finite set of axiom schema. Notice that the inductive axioms for PA are given as an axiom schema. Expanding on the comments from class on the ability to recognize in a primitive recursive fashion an inductive axiom in PA, describe an algorithm for determining if a given number is the Gödel number of an axiom of a theory described by finitely many axiom schema.

3. Formally write out the $\Sigma_1$-formula expressing $y = x!$ in $L_A$. You may use the types of shorthand that we used in class but be sure that it is clear that the formula you write down could be turned into an $L_A$ formula.

4. Not everything is undecidable! Let $\mathbf{Q} = (Q, +, -, 0)$ where $Q$ is the rational numbers, $+$ is the usual addition function on $Q$, $-$ is a unary function indicating the additive inverse and 0 is a constant interpreted as 0. The following steps show that the theory of $\mathbf{Q}$ is decidable.

(a) The following universally quantified formulas hold in **Q**:

   i. $x + 0 = x$;

   ii. $x + (-x) = 0$;

   iii. $(x + y) + z = x + (y + z)$;

   iv. $x + y = y + x$

   Show that $-(-x) = x$ and $-(x + y) = -x + (-y)$ follow from these axioms.

(b) Using the abbreviation $nx$ for $x + (x + (x \ldots) \ldots)$ where the $x$ is repeated $n$ times for positive $n$ and $-nx$ for $-x + (-x + \ldots) \ldots)$ where $-x$ is repeated $n$ times, show that modulo the axioms above, every term in the language of **Q** is equivalent to one of the form $n_1x_1 + (n_2x_2 + \ldots n_kx_k) \ldots)$ for integers $n_1, \ldots, n_k$.

(c) Conclude that modulo the axioms given, every atomic formula is equivalent to one of the form (dropping the brackets):

$$n_1x_1 + n_2x_2 + \ldots n_kx_k = 0.$$

(d) Another list of axioms for this theory are the divisibility axioms. Notice that for every positive integer $n$, the following holds in **Q**:

$$\forall y \exists ! x (nx = y).$$

Show that for every rational number $r = p/q$, the axioms listed so far prove that there is a function $rx$ defined by

$$y = rx \text{ iff } px = qy.$$

(e) The final axioms for **Q** are sentences that say **Q** has more than $n$ elements for every $n \in \mathbf{N}$ (one sentence for every $n$). Use this to show that every formula is equivalent to a boolean combination of atomic formulas. Hint: do this by induction on the formation of the formula. Everything except the quantifier step is easy. Notice two additional things: first you only have to consider the case of an existential quantifier, and second, since every boolean combination of atomic formulas is equivalent to a disjunction of conjunctions of atomic and negated atomic formulas, you only have to consider conjunctions of atomic and negated atomic formulas.

(f) Conclude that the axioms listed for **Q** are complete and that this theory is decidable.