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Recall: Gödel devised a clever trick for coding the symbols in L_A as numbers

The key idea is to enumerate all the symbols in a decidable, one-to-one fashion starting from the number 1

e.g. you could enumerate them as:

$$\begin{array}{ccccccccccccccc} (&) & \wedge & \vee & \neg & \rightarrow & \forall & \exists & + & \times & S & 0 & 1 & x_0 & x_1 & \dots \\ | & | & | & | & | & | & | & | & | & | & | & | & | & | & | & \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & \dots \end{array}$$

So to every symbol s, we assign a unique number $\ulcorner s \urcorner > 0$ and this is done decidably

If we have a string of symbols $s_0 s_1 s_2 \dots s_n$ then we will code the string as:

$$2^{\ulcorner s_0 \urcorner} 3^{\ulcorner s_1 \urcorner} 5^{\ulcorner s_2 \urcorner} \dots \pi(i)^{\ulcorner s_i \urcorner} \dots \pi(n)^{\ulcorner s_n \urcorner}$$

where $\pi(i)$ is the i^{th} prime and $\ulcorner s_0, \dots, s_n \urcorner$ is its Gödel number

Remark: There are several things to remember:

- $\pi(i)$ is a p.r. function
- Since we don't use 0 as a number for any s, we know that the number of primes used in its factorization is $n+1$
- We know and can tell what the exponent is of any prime in this factorization by using the p.r. function $\exp(n,i)$ — the function which outputs the exponent of $\pi(i)$ in the prime factorization of n

Definition: If we are given a term τ then it is just a string of symbols and so it has a Gödel number $\ulcorner \tau \urcorner$

Example: If τ is $SS0$, then $\ulcorner SS0 \urcorner$ is:

$$2^{\ulcorner S \urcorner} 3^{\ulcorner S \urcorner} 5^{\ulcorner 0 \urcorner}$$

The exponents which minimize this Gödel number (just for illustrative purposes) is $\ulcorner 0 \urcorner = 1$ and $\ulcorner S \urcorner = 2$, which means that $SS0$ (which we commonly know as the number 2) is at least

$$2^2 3^2 5^1 = 180$$

You can imagine that as the strings get larger, this number blows up very fast

Definition: Similarly, for any formula φ, we can view it as a string of symbols and find its Gödel number $\ulcorner \varphi \urcorner$

Observation: We can do a number of things algorithmically and in a p.r. fashion

Example: Given a number n, we can determine if n codes a string of symbols

This entails seeing if the prime factorization of n involves <u>all</u> the primes from 2 up to some given prime and then writing

$$n = 2^{k_0} 3^{k_1} \cdots \pi(m)^{k_m}$$

We then decode $k_0, \ldots, k_m$ as a string of symbols $s_0, \ldots, s_m$ and now we have a decidable process for determining if the string is a term/formula/neither

Remark: 10 cannot code a string of symbols since $10 = 2^1 3^0 5^1$