Assignment 2, Math 3EE3
Due Feb. 5 in class

(1) Fix a ring R and let S be the set of all functions from R to R.
  (a) If we put addition on S as follows: for $f, g \in S$
  $$(f+g)(r) = f(r) + g(r) \text{ for all } r \in R$$
  and multiplication defined by multiplication of functions, show S is a ring.
  (b) Show that the set of polynomially defined functions P contained in S is a subring of S.
  (c) Show that P and $R[x]$ are not necessarily the same; for instance when R is finite.

(2) Show that over the ring $M_2(C)$, 2 x 2 matrices over the complex numbers, the equation $x^2 = I$ has infinitely many solutions.

(3) Let the set of formal power series over a field F be defined as
$$F[[X]] = \{\sum_{i=0}^{\infty} f_i X^i : f_i \in F\}$$
For any $f = \sum_{i=0}^{\infty} f_i X^i$, $g = \sum_{i=0}^{\infty} g_i X^i$ define addition and multiplication on $F[[X]]$ by:
$$f + g = \sum_{i=0}^{\infty} (f_i + g_i) X^i,$$
$$fg = \sum_{i=0}^{\infty} (\sum_{j=0}^{i} f_j g_{i-j}) X^i.$$
Prove that $F[[X]]$ is an integral domain.

(4) Fix an abelian group $(G, +)$. A group homomorphism from G to G is called an endomorphism. Let $End(G)$ be the set of all endomorphisms of G.
  (a) Show that the sum of two endomorphisms is an endomorphism.
  (b) Show that the composition of two endomorphims is an endomorphism.
  (c) Show then that $End(G)$ with addition and composition is a ring with unity.

(5) The goal of this question is to construct the real numbers algebraically. Start with the ring Q^N, sequences of rational numbers, and let C be the subset of Cauchy sequences i.e. those sequences $(a_i : i \in N)$ such that for every k there is a number M such that if $i, j \geq M$ then $|a_i - a_j| \leq 1/k$. Show that C forms a subring of Q^N.

Define a function $\varphi : C \to \mathbb{R}$ by $\varphi((a_i : i \in N)) = \lim_{i \to \infty} a_i$. Show that φ is a surjective homomorphism and identify its kernel I. This shows that $\mathbb{R} \cong C/I$.