Solutions to Assignment 5

1 a) Let's compute e^{xN} since that is what we need for b).

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & 0 & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ & & & & 0 \\ 0 & & & & 0 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & & 0 \\ & 0 & 0 & 0 & 1 & \\ & & \ddots & & & \vdots \\ 0 & & & & & 1 \\ & & & & & 0 \\ & & & & & 0 \end{pmatrix} \text{ etc.}$$

$$N^n = 0 \quad \text{so} \quad e^{xN} = \begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ & 1 & x & x^2 & \vdots \\ & & \ddots & \ddots & x^2 \\ & 0 & & & x \\ & & & & 1 \end{pmatrix}$$

b) The general solution is $y = e^{xA} y_0$ and

$$e^{xA} = e^{x\lambda I} e^{xN}$$

$$= \begin{pmatrix} e^{\lambda x} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda x} \end{pmatrix} \begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ & 1 & \ddots & & \vdots \\ & & & \ddots & x \\ 0 & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda x} & xe^{\lambda x} & \cdots & x^{n-1} e^{\lambda x} \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ 0 & & & e^{\lambda x} \end{pmatrix}$$

2. We prove this as in the real case by induction on $\dim(V)$ using Gram-Schmidt.

So we can reduce to the case where $\dim(V) = n$, $W \subseteq V$, $\dim(W) = n-1$ and W has an orthogonal basis $w_1 \cdots w_{n-1}$. Extend this basis to V by adding a lin. indep v.

Let $v_i = \frac{\langle v, w_i \rangle}{\|w_i\|^2} w_i$ and $w_n = v - (v_1 + \cdots + v_{n-1})$

Computing $\langle w_n, w_i \rangle$ for $i < n$: Since $\langle w_j, w_i \rangle = 0$ if $i \neq j$ we get

$$\langle w_n, w_i \rangle = \langle v, w_i \rangle - \frac{\langle v, w_i \rangle}{\|w_i\|^2} \langle w_i, w_i \rangle = 0.$$

Now $w_n \notin W$ since $v \notin W$ and so $w_n \neq 0$ and it is orthogonal to w_i for all i so $w_1 \cdots w_n$ forms an orthogonal basis for V.

Notice we did not use the fact that the scalar product was hermitian anywhere.

3. Define $e : V \to V^{**}$ by $e(v)(\varphi) = \varphi(v)$ for all $\varphi \in V^*$. For $\varphi, \psi \in V^*$ and $\lambda \in K$

$$e(v)(\varphi+\psi) = (\varphi+\psi)(v) = \varphi(v)+\psi(v) = e(v)(\varphi) + e(v)(\psi)$$

$$e(v)(\lambda\varphi) = \lambda\varphi(v) = \lambda e(v)$$

so $e(v) \in V^{**}$. Now to see that e itself is a linear map: for $v, w \in V$ and any $\varphi \in V^*$

$$e(v+w)(\varphi) = \varphi(v+w) = \varphi(v) + \varphi(w)$$
$$= e(v)(\varphi) + e(w)(\varphi)$$

so $e(v+w) = e(v) + e(w)$.

For $v \in V$, $\lambda \in K$ and $\varphi \in V^*$

$$e(\lambda v)(\varphi) = \varphi(\lambda v) = \lambda\varphi(v) = \lambda e(v).$$

So e is a linear map. If $e(v) = 0$ then $\varphi(v) = 0$ for all $\varphi \in V^*$. If $v \neq 0$ then we can find <u>some</u> φ s.t. $\varphi(v) \neq 0$ (for instance, suppose $v \in S$, S a basis for V - then the dual basis for V^* contains a φ s.t. $\varphi(v) = 1$ and $\varphi(w) = 0$ if $w \in S$, $w \neq v$).

So $v = 0$ and e is 1-1.

4. a) Suppose λ is an eigenvalue for A; let V_λ = the λ-eigenspace for A.

Claim: V_λ is A^*-invariant.

Pf/ If $w \in V_\lambda$ then $Aw = \lambda w$ so $A^*Aw = \lambda A^*w$ and $A^*A = AA^*$ so $AA^*w = \lambda A^*w$ i.e. $A^*w \in V_\lambda$.

Now pick any $v \in V_\lambda$ which is an eigenvector for A^* and it will also be one for A. Suppose $A^*v = \mu v$

b) Suppose $w \in v^\perp$. Then

$$\langle Aw, v\rangle = \langle w, A^*v\rangle = \langle w, \mu v\rangle = 0 \quad \text{and}$$
$$\langle A^*w, v\rangle = \langle w, Av\rangle = \langle w, \lambda v\rangle = 0$$

so $v^\perp$ is A and A^*-invariant.

c) By induction on $\dim(V)$, since $\dim(v^\perp) < \dim(V)$, we can find a basis of orthogonal eigenvectors $v_2, \ldots v_n$ for both A and A^*.

Putting this together with v we get

$v, v_2 \ldots v_n$, a basis of orthogonal eigenvectors for A (and A^*).