Solutions to Assignment #2

1\. a) An n-dim. v. sp. over $\mathbb{Z}_2$ is bijective with $\mathbb{Z}_2^n$ so the number of vectors is 2^n.

b) and c)! For an $n \times n$ matrix to be invertible, the 1^{st} column cannot be 0 so there are $2^n - 1$ choices. The 2^{nd} column cannot be a mult. of the first and so over $\mathbb{Z}_2$, this leaves $2^n - 2$ possibilities.
For the 3^{rd} column, it cannot lie in the space gen. by the first two so there are $2^n - 2^2$ possibilities.

For the i^{th} column, it cannot lie in the subspace generated by the first $i-1$ so there are $2^n - 2^{i-1}$ choices.

This means there are $(2^n-1)(2^n-2^2)\cdots(2^n-2^{i-1})\cdots(2^n-2^{n-1})$ invertible $n \times n$ matrices over $\mathbb{Z}_2$.

2\. Suppose that $u_1 \cdots u_m$ is a basis for U, $w_1 \cdots w_n$ is a basis for W.

Claim: $(u_1, 0), \cdots (u_m, 0), (0, w_1) \cdots (0, w_n)$ is a basis for $U \times W$.

Lin indep: Suppose $\lambda_1(u_1, 0) + \cdots + \lambda_m(u_m, 0) + \mu_1(0, w_1) + \cdots + \mu_n(0, w_n) = (0, 0)$.

Then $\lambda_1 u_1 + \cdots + \lambda_m u_m = 0$ and $\mu_1 w_1 + \cdots + \mu_n w_n = 0$ and so $\lambda_1 = \cdots = \lambda_m = 0$, $\mu_1 = \cdots = \mu_n = 0$ since $u_1 \cdots u_m$, $w_1 \cdots w_n$ are lin. indep.

Span: if $(u, w) \in U \times W$ then

$u = \lambda_1 u_1 + \cdots + \lambda_m u_m$, $w = \mu_1 w_1 + \cdots + \mu_n w_n$ for some $\lambda_1 \cdots \lambda_m, \mu_1 \cdots \mu_n \in K$.

So $(u, w) = \lambda_1 (u_1, 0) + \cdots + \lambda_m (u_m, 0) + \mu_1 (0, w_1) + \cdots + \mu_n (0, w_n)$

3. Suppose $\dim(U) = m$, $\dim(W) = n$ and $\dim(U \cap W) = k$

Pick a basis $v_1 \cdots v_k$ for $U \cap W$. Extend this basis to one for U, say $u_{k+1} \cdots u_m$ and extend $v_1 \cdots v_k$ to a basis for W, say $w_{k+1} \cdots w_n$.

Claim: $v_1 \cdots v_k, u_{k+1} \cdots u_m, w_{k+1} \cdots w_n$ is a basis for $U + W$.

Span: This is the easier of the two things we need to check. If $u + w \in U + W$, then $u \in \text{span}(v_1 \cdots v_k\ u_{k+1} \cdots u_m)$ and $w \in \text{span}(v_1 \cdots v_k, w_{k+1}, \cdots w_n)$ so $u + w \in$ span of our suggested basis.

Lin. indep: Suppose

$$\delta_1 v_1 + \cdots \delta_k v_k + \lambda_{k+1} u_{k+1} + \cdots + \lambda_m u_m + \mu_{k+1} w_{k+1} + \cdots + \mu_n w_n = 0$$

We have

$$(*) \quad \underbrace{\delta_1 v_1 + \cdots + \delta_k v_k + \lambda_{k+1} u_{k+1} + \lambda_m u_m}_{\in U} = \underbrace{-\mu_{k+1} w_{k+1} - \cdots - \mu_n w_n}_{\in W}.$$

So $-\mu_{k+1} w_{k+1} \cdots - \mu_n w_n \in U \cap W$ which means it can be written as a linear comb. of $v_1 \cdots v_k$.

Say $-\mu_{k+1} w_{k+1} \quad -\mu_n w_n = \alpha_1 v_1 + \cdots + \alpha_k v_k$.

But $v_1 \cdots v_k, w_{k+1} \cdots w_n$ are lin indep. so all of these scalars are 0.

Going back to (*) then we see that

$\delta_1 v_1 + \cdots + \delta_k v_k + \lambda_{k+1} u_{k+1} + \cdots + \lambda_m u_m = 0$ and

$v_1 \cdots v_k \, u_{k+1} \cdots u_m$ is lin. indep so all of these scalars are 0 as well.

We conclude then that $\dim(U+W) = m+n-k$.

4. If $C = (c_{ij})$, $D = (d_{ij})$ are two $n \times n$ matrices then the i^{th} entry on the diagonal of CD is

$c_{i1} d_{1i} + \cdots + c_{in} d_{ni}$

So if $A = (a_{ij})$, $B = (b_{ij})$ are $n \times n$ matrices then

$tr(AB) = a_{11} b_{11} + \cdots + a_{1n} b_{n1} + a_{21} b_{12} + \cdots a_{2n} b_{n2} + \cdots$

$+ a_{n1} b_{1n} + \cdots + a_{nn} b_{nn}$.

Whereas,

$$tr(BA) = b_{11}a_{11} + \cdots + b_{1n}a_{n1} + b_{21}a_{12} + \cdots + b_{2n}a_{n2} + \cdots b_{n1}a_{1n} + \cdots + b_{nn}a_{nn}$$

which are seen to be equal by reordering the terms.