Solution to Test 2

1. Suppose that $f(x) = a_n x^n + \cdots + a_0$ and $g(x) = b_m x^m + \cdots + b_0$.

Then $f(A) = a_n A^n + \cdots + a_0 I$ and
$g(A) = b_m A^m + \cdots + b_0 I$.

$$f(A)g(A) = a_n A^n b_m A^m + \cdots + a_i A^i b_j A^j + \cdots + a_0 I b_0 I$$

$$= a_n b_m A^{n+m} + \cdots + a_i b_j A^{i+j} + \cdots + a_0 b_0 I.$$

$$= (fg)(A).$$

2. a) If A is an $n \times n$ matrix over K then the minimal polynomial for A is the polynomial $f(x) \in K[x]$ of least degree such that $f(A) = 0$ and the leading coeff. is 1.

b) By the Cayley-Hamilton Theorem, if $g(x)$ is the characteristic poly. of A then $g(A) = 0$. If f is the minimal poly. of A then also $f(A) = 0$.

Using the division algorithm, write $g = qf + r$ with $\deg(r) < \deg(f)$. Then $g(A) = q(A)f(A) + r(A)$ i.e. $r(A) = 0$. Since f is the minimal poly., $r = 0$ i.e. f divides g.

3 a) W is said to be A-invariant if for every $w \in W$, $Aw \in W$.

b) Suppose that $w \in \ker(f(A))$ i.e. $f(A)w = 0$. We need to see that $f(A)(Aw) = 0$. But A and $f(A)$ commute so $f(A)(Aw) = A(f(A))w = 0$.

4. By the Jordan canonical form theorem, the similarity type is determined by the set of Jordan blocks.

For the characteristic poly. $(x-1)(x-2)^2(x-3)^3$ there are 3 types of blocks: ones for the eigenvalues 1, 2 and 3. There is only one possible block for $\lambda = 1$ - 1×1. For $\lambda = 2$ we could have $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

For $\lambda = 3$, we could have $\begin{pmatrix} 3 & & 0 \\ & 3 & \\ 0 & & 3 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & \\ & 0 & 3 \end{pmatrix}$

or $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Altogether then there are $1 \times 2 \times 3 = 6$ similarity types.

5. a) We are asked to solve $Ax = \lambda x$ where

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

which in reduced row echelon form looks like:

$$\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

The solution is then

$x = ke_1$ for $k \in K$.

b) Suppose $A = P^{-1}BP$ with P invertible and $v_1, \ldots, v_k$ is a basis for the λ-eigenspace of A.

$PA = BP$ and $Av_i = \lambda v_i$ so $PAv_i = \lambda Pv_i = BPv_i$.

This means that Pv_i is in the λ-eigenspace for B. Moreover, since P is invertible, $Pv_1, \ldots Pv_k$ are lin. indep. It follows that $\dim(\lambda\text{-eigenspace of } B) \geq \dim(\lambda\text{-eigenspace of } A)$. But by symmetry, since $B = PAP^{-1}$, we also get the reverse inequality and so the dimensions are the same.

c) By a), the dimension of the λ-eigenspace for a λ-Jordan block is 1 and by the Jordan canonical form theorem says that every matrix is similar to one in Jordan canonical form, the dimension of the λ-eigenspace for a matrix is the same as the number of Jordan blocks with eigenvalue λ in the JCF.