Math 2S3, Practice test

Please write complete answers to all of the questions in the test booklet provided. Partial credit may be given for your work. Unless otherwise noted, you need to justify your solutions in order to receive full credit. Please be sure to include your name and student number on all sheets of paper that you hand in.

1. (5 marks) Assume V is a complex vector space. We said that a multilinear function $f : V \times V \to W$ is alternating if for all $u, v \in V$,

   $$f(u,v) = -f(v,u)$$

   or alternatively for every $v \in V$,

   $$f(v,v) = 0$$

   Show that these two definitions are in fact equivalent.

2. (5 marks)

   (a) Show that if λ is an eigenvalue for A then the eigenspace V_λ is invariant for A i.e. for all $v \in V_\lambda, Av \in V_\lambda$.

   (b) Use part (a) to show that if the only subspaces of V which are invariant for A are 0 and V then A has only one eigenvalue.

   (c) Use part (b) to show that if the only invariant subspaces for A are 0 and V then $A = \lambda id_V$.

3. (5 marks) Suppose that f is a polynomial, $A : V \to V$ is a linear operator and $B : V \to V$ is an invertible linear operator. Show that $f(B^{-1}AB) = B^{-1}f(A)B$.

4. (5 points) Suppose that A is a linear operator on a finite-dimensional K-vector space and $f \in K[x]$ is of least degree such that $f(A) = 0$. Use the Cayley-Hamilton theorem to show that f divides the characteristic polynomial of A.

5. (5 points) Up to similarity, how many 9×9 complex matrices are there with characteristic polynomial $(x-1)^4(x-2)^3(x-3)^2$?