Math 2S3

Dr. Bradd Hart Apr. 21, 2014

This examination is 3 hours in length. Attempt all 10 questions. The total number of available points is 50. Marks are indicated next to each question. Write your answers in the booklets provided. You must show your work to get full credit. **Good Luck.**

1. (5 points) Which of the following is a field? (no explanation required)

   (a) Q, the rationals

   (b) $Q[x]$, the set of polynomials with rational coefficients

   (c) $Q[\sqrt{2}] = \{p(\sqrt{2}) : p \in Q[x]\}$

   (d) $M_2(Q)$, 2×2 matrices with rational entries

2. (5 points) Which of the following statements are true? If the statement is false, give a counter-example.

   (a) If S is a basis for a vector space V then S generates V.

   (b) If S generates a vector space V then S is a basis for V.

   (c) If V is a vector space with a scalar product and S is a basis for V then S is orthogonal.

3. (5 points) Prove that if V is a finite-dimensional K-vector space then $V \cong V^*$ where V^* is the dual space of V i.e. the set of linear functionals from V into K.

4. (5 points) Show that if V is a complex vector space with a non-degenerate hermitian form and A is an hermitian operator on V then any eigenvalue of A is real.

5. (5 points) Suppose that S, U and W are subspaces of a K-vector space V. Prove that

$$S \cap \langle U \cup W \rangle = \langle \langle S \cap U \rangle \cup \langle S \cap W \rangle \rangle$$

   where $\langle X \rangle$ means the subspace generated by X.

6. (5 points)

   (a) Compute e^{xA} where A is the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   (b) Solve $y' = Ay$ where y is

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

continued on the other side

7. (5 points) Suppose that V is a K-vector space and A is a linear operator on V.

   (a) If $f \in K[x]$ explain what the linear operator $f(A)$ is.

   (b) If $f, g \in K[x]$, explain why $f(A)g(A) = (fg)(A)$.

8. (5 points) Suppose that A is a linear operator on a finite-dimensional K-vector space and $f \in K[x]$ is of least degree such that $f(A) = 0$. Use the Cayley-Hamilton theorem to show that f divides the characteristic polynomial of A.

9. (5 points) Suppose that K is a field.

   (a) If $I \subseteq K[x]$, define what it means for I to be an ideal.

   (b) Show that if $I \subseteq K[x]$ is an ideal then it is generated by a single polynomial.

10. (5 points) Up to similarity, how many 9×9 complex matrices are there with characteristic polynomial $(x-1)^4(x-2)^3(x-3)^2$?