Matrices with complex entries

- From now on, unless it is explicitly said otherwise, matrices will be assumed to have complex entries.
- All basic linear algebra - linear equations with complex coefficients, matrix multiplication and addition, determinant calculations - work exactly the same over the complex numbers as they do over the reals.
- In particular, a square matrix is invertible iff its determinant is non-zero.
- The biggest advantage of using the complex numbers is that characteristic polynomials will always have roots so every square complex matrix has at least one eigenvalue.

Vector Space Axioms

Suppose V is a set together with the operations $+$ and multiplication by scalars (real numbers). Then we call V a (real) vector space if the following axioms are satisfied:

1. If u and v are objects in V, then $u + v$ is in V;
2. For all u and v in V, $u + v = v + u$;
3. For all u, v and w in V, $u + (v + w) = (u + v) + w$;
4. There is an object 0 in V such that for all u in V, $0 + u = u$;
5. For all u in V, there is an object $-u$ in V such that $u + (-u) = 0$;

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5. For all u in V, there is an object $-u$ in V such that $u + (-u) = 0$;
6. For any scalar k and any u in V, ku is in V;
7. For any scalar k and u, v in V, $k(u + v) = ku + kv$;
8. For scalars k and m, and any u in V, $(k + m)u = ku + mu$;
9. For scalars k and m, and any u in V, $k(mu) = (km)u$; and
10. For all u in V, $1u = u$.

Subspaces

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Theorem

A subset W of a vector space V is a subspace of V if

1. *W is closed under $+$ i.e. if u and v are in W then $u + v$ is in W, and*
2. *W is closed under scalar multiplication i.e. if k is a scalar and u is in W then ku is in W.*

Linear independence

Definition

If $S = \{v_1, v_2, \ldots, v_r\}$ is a non-empty set of vectors such that the only solution for scalars $k_1, k_2, \ldots, k_r$ of the equation

$$k_1 v_1 + k_2 v_2 + \ldots + k_r v_r = 0$$

is $k_1 = k_2 = \ldots = k_r = 0$ then S is said to be linearly independent. Otherwise, S is linearly dependent.

Basis and Dimension

Definition

If V is a vector space and $S = \{v_1, v_2 \dots, v_n\}$ is a set of vectors in V then S is said to be a basis for V if

1. S is linearly independent and
2. S spans V.

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Definition

A vector space V is called finite-dimensional if it has a finite basis. Otherwise it is called infinite-dimensional.

Theorem (4.5.1)

If V is a finite-dimensional vector space then all bases for V have the same number of vectors.