Quadratic forms

Definition

Suppose $x_1, x_2, \ldots, x_n$ are variables.

- A monomial of degree 2 is a function of the form $x_i x_j$ for some i, j such that $1 \leq i, j \leq n$.
- A quadratic form in the variables $x_1, x_2, \ldots, x_n$ is a linear combination of monomials of degree 2.

Illustrative examples

- If A is any invertible matrix then $\langle x, y \rangle = Ax \cdot Ay$ is an inner product and $\langle x, x \rangle$ is a quadratic form.
- In general, if B is a symmetric matrix then $x^T Bx$ is a quadratic form and any quadratic form is of this kind.

Positive and negative definite

Definition

Suppose that q is a quadratic form and $q = x^T Ax$ for some symmetric matrix A. q and A are called

1. positive definite if $q(x) > 0$ for all $x \neq 0$,
2. negative definite if $q(x) < 0$ for all $x \neq 0$, and
3. indefinite if $q(x)0$ can be both positive and negative.

Theorem

For a symmetric matrix A, A is positive (negative) definite iff all its eigenvalues are positive (negative).

Principal submatrices

Definition

If A is an $n \times n$ matrix then the principal submatrices of A are the n square matrices formed by entries in the first r rows and columns as r varies from 1 to n.

Theorem

For a symmetric matrix A, A is positive definite iff the determinants of all its principal submatrices are positive. A is negative definite if $-A$ is positive definite.

Theorem (7.3.1)

If A is an $n \times n$ symmetric matrix then if P orthogonally diagonalizes A i.e. $D = P^T AP$ for a diagonal matrix D with diagonal $\lambda_1, \ldots, \lambda_n$, and $x = Py$ for two n-tuples of variables x and y then

$$x^T Ax = y^T Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2.$$

The change of variables in the theorem, $x = Py$, is called an orthogonal change of variables.

Quadratic equations and conic sections

- A quadratic equation is one of the form

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

  where at least one of a, b or c is not zero.
- There are three types of conic sections in standard position:
  - Ellipses and circles:

$$\frac{x^2}{k^2} + \frac{y^2}{l^2} = 1$$

  - Hyperbolas:

$$\frac{x^2}{k^2} - \frac{y^2}{l^2} = 1 \text{ or } \frac{y^2}{k^2} - \frac{x^2}{l^2} = 1$$

  - Parabolas:

$$y = kx^2 \text{ or } x = ky^2$$

Quadratic equations as conic sections

- Problem: How do we understand a quadratic equation as the graph of a conic section in the plane?
- Two parts of the solution:
  - The conic may not be centered at the origin: we can tell this if there is no "cross-term" i.e. no xy term in the equation. Solution: complete the square to determine how translated the conic is.
  - It may be rotated. You will be able to tell this if there is a cross-term present. Solution: Orthogonally diagonalize the associated quadratic form and change variables to see what conic section you have.
  - For the quadratic equation

    $$ax^2 + 2bxy + cy^2 + dx + ey + f = 0,$$

    the associated quadratic form is

    $$ax^2 + 2bxy + cy^2.$$