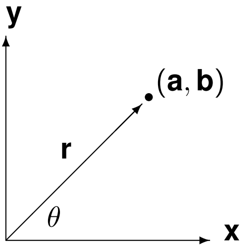
How to picture complex numbers: the complex plane



- Associate to the complex number $z = a + bi$ the point on the plane (a, b). $r = \sqrt{a^2 + b^2}$ is called the modulus of z and written $|z|$.
- We saw that $z \cdot \bar{z} = |z|^2$.
- θ is called an argument for $a + bi$ and is only determined up to multiples of 2π.
- $a = r\cos(\theta)$ and $b = r\sin(\theta)$ so $z = r(\cos(\theta) + i\sin(\theta))$.

Exponentiation and complex numbers

We define exponentiation for complex numbers via the formula

$$e^{a+ib} = e^a(\cos(b) + i\sin(b))$$

In particular,

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

so if $z = a + ib$, $r = |z|$ and θ is an argument for z then

$$z = re^{i\theta}$$

Useful tricks with exponentiation

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$\begin{aligned}
z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} \\
&= r_1 r_2 (\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\
z_1/z_2 &= r_1 e^{i\theta_1}/r_2 e^{i\theta_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \\
&= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \\
z_1^n &= r_1^n e^{in\theta_1}
\end{aligned}$$

This says that when we multiply z_1 and z_2 we multiply their moduli and add their arguments; when we divide we divide we divide the moduli and subtract the arguments.

Useful tricks with exponentiation, cont'd

Roots of unity

It is often useful to know all the solutions to $x^n = 1$. They are

$$e^{\frac{2k\pi i}{n}} \text{ for } k = 0, 1, \ldots n - 1.$$

Useful tricks with exponentiation, cont'd

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Wonderful formula

$$e^{i\pi} + 1 = 0$$

Matrices with complex entries

- From now on, unless it is explicitly said otherwise, matrices will be assumed to have complex entries.
- All basic linear algebra - linear equations with complex coefficients, matrix multiplication and addition, determinant calculations - work exactly the same over the complex numbers as they do over the reals.
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- In particular, a square matrix is invertible iff its determinant is non-zero.
- The biggest advantage of using the complex numbers is that characteristic polynomials will always have roots so every square complex matrix has at least one eigenvalue.