Kernel and Range

Definition

If $T : V \to W$ is a linear transformation then the *kernel* of T, written $\ker(T)$ is the set of all $v \in V$ such that $T(v) = 0$. The range of T, written $R(T)$, is the set of all vectors in W of the form $T(v)$ for some $v \in V$.

Theorem

If $T : V \to W$ is a linear transformation then $\ker(V)$ is a subspace of V and $R(T)$ is a subspace of W.

Rank and Nullity

Definition

If $T : V \to W$ is a linear transformation then the dimension of $R(T)$ is called the rank of T and written $\text{rank}(T)$. The dimension of the $\ker(T)$ is called the nullity of T and written $\text{nullity}(T)$.

Theorem

If A is a matrix then $\text{nullity}(T_A) = \text{nullity}(A)$ and $\text{rank}(T_A) = \text{rank}(A)$.

Theorem

If $T : V \to W$ is a linear transformation from an n-dimensional vector space V then $\text{rank}(T) + \text{nullity}(T) = n$.

One-to-one linear transformations

Definition

A linear transformation $T : V \to W$ is said to be *one-to-one* if T sends distinct vectors in V to distinct vectors in W. Said another way, for all $v, w \in V$, if $T(v) = T(w)$ then $v = w$.

Theorem

If $T : V \to W$ is a linear transformation then the following are equivalent:

1. *T is one-to-one.*
2. $\ker(T) = \{0\}$.
3. *nullity(T) = 0.*

Theorem

If V is finite-dimensional and $T : V \to V$ is a linear operator then T is one-to-one iff $R(T) = V$ (the range of T is V).

Inverse linear transformations

Definition

If $T : V \to W$ is a one-to-one linear transformation, we define the *inverse* of T, and write T^{-1}, by $T^{-1} : R(T) \to V$ by

$$T^{-1}(w) = v \text{ iff } T(v) = w$$

Theorem

If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations then

1. $T_2 \circ T_1$ *is one-to-one, and*
2. $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

Surjections and isomorphisms

Definition

- If $T : V \to W$ is a linear transformation then we say that T is *surjective* (or onto) if the range of T is W i.e. $R(T) = W$. T is said to be a surjection.
- If $T : V \to W$ is called a *bijection* if it is one-to-one and onto (surjective). We also call a bijection an *isomorphism*. T is said to be bijective and V and W are said to be isomorphic.

Theorem

If V and W are vector spaces with different dimensions then V and W are not isomorphic.

Surjections and isomorphisms, cont'd

Theorem

If V is an n-dimensional vector space then V is isomorphic to R^n if V is a real vector space and is isomorphic to C^n if it is a complex vector space.

Corollary

If V and W have the same finite dimension then V and W are isomorphic.