Linear Transformations

Definition

If V and W are vector spaces and $T : V \to W$ is a function from V to W then we say that T is a *linear transformation* if for all $u, v \in V$ and scalars c,

1. $T(u + v) = T(u) + T(v)$, and
2. $T(cu) = cT(u)$.

In the case where $V = W$ and $T : V \to V$, we call T a *linear operator*.

Properties of Linear Transformations

Theorem

If $T : V \to W$ is a linear transformation then

1. $T(0) = 0$
2. $T(-v) = -T(v)$ *for all* $v \in V$
3. $T(v - w) = T(v) - T(w)$ *for all* $v, w \in V$

Action on a basis

Very Important Fact

A linear transformation is completely determined by its action on a basis. That is, if $T : V \to W$ is a linear transformation and $v_1, v_2, \ldots, v_n$ is a basis for V then, since any $v \in V$ is of the form

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

then

$$T(v) = c_1 T(v_1) + c_2 T(v_2) + \ldots + c_n T(v_n)$$

so T is determined by the values $T(v_1), T(v_2), \ldots, T(v_n)$.

Composition of Linear Transformations

Theorem

If U, V and W are vector spaces and $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations then the composition of T_2 with T_1, $T_2 \circ T_1$, defined by

$$(T_2 \circ T_1)(u) = T_2(T_1(u))$$

is a linear transformation from U to W.

Kernel and Range

Definition

If $T : V \to W$ is a linear transformation then the *kernel* of T, written $\ker(T)$ is the set of all $v \in V$ such that $T(v) = 0$. The range of T, written $R(T)$, is the set of all vectors in W of the form $T(v)$ for some $v \in V$.

Theorem

If $T : V \to W$ is a linear transformation then $\ker(V)$ is a subspace of V and $R(T)$ is a subspace of W.