Projection as best approximation

Theorem

Suppose that W is a finite-dimensional subspace of an inner product space V. Then for any $u \in V$, $proj_W u$ is the closest vector in W to u; that is, if $w \in W$ is any vector other than $proj_W u$ then

$$||u - w|| > ||u - proj_W u||$$

Normal equations

- It is not always possible to solve $Ax = b$. What about finding x such that $\|Ax - b\|$ is minimized?
- If W is the columnspace of A then this means finding x such that $Ax = \text{proj}_W b$.
- We want to find x such that $Ax - b$ is orthogonal to W.
- Since to be orthogonal to the columnspace of A means being in the nullspace of A^T, this means we want to solve

  $$A^T Ax = A^T b.$$

  These are the normal equations.
- If the columns of A are linearly independent then

  $$\text{proj}_W b = A(A^T A)^{-1} A^T b.$$

Continuous functions on $[0, 2\pi]$

- We have seen that $C[0, 2\pi]$ is a vector space with the integral inner product.
- With respect to this inner product, if W is a finite-dimensional subspace of $C[0, 2\pi]$, then the distance from any f to W is given by $\|f - \text{proj}_W(f)\|$.
- We have a known list of orthogonal functions in $C[0, 2\pi]$:

$$1, \sin(x), \sin(2x), \ldots \cos(x), \cos(2x), \ldots$$

Fourier series

- Let W_n be the subspace generated by

$$1, \sin(x), \ldots, \sin(nx), \cos(x), \ldots, \cos(nx)$$

inside $C[0, 2\pi]$.

- Since the generators of each W_n form an orthogonal set, they are linearly independent and it is easy to compute the projection onto W_n.
- For any $f \in C[0, 2\pi]$ we compute

$$a_0 = \frac{\langle f, 1\rangle}{\|1\|^2}, a_k = \frac{\langle f, \sin(kx)\rangle}{\|\sin(kx)\|^2} \text{ and } b_k = \frac{\langle f, \cos(kx)\rangle}{\|\cos(kx)\|^2}.$$

for all $k \geq 1$.

Main Theorem

Theorem

If $f \in C[0, 2\pi]$ *then* $f(x)$ *converges to*

$$a_0 + a_1 \sin(x) + a_2 \sin(2x) + \ldots + b_1 \cos(x) + b_2 \cos(2x) + \ldots$$

with respect to $\|\cdot\|$.

Example

If W is the subspace generated by

$$1, \sin(x), \sin(2x), \ldots, \cos(x), \cos(2x), \ldots$$

then by the Main Theorem, $W^\perp = 0$. But $0^\perp$ is all of $C[0, 2\pi]$. W is not all of $C[0, 2\pi]$ since $x \not\in W$ so we have an example of $(W^\perp)^\perp \neq W$.

Test 1

Test Average and Median: 14

Test Score	Number
20 - 25	20
17 - 19	32
14 - 16	31
11 - 13	38
Below 10	34
DNW	22