Triangular matrices

Definition

Suppose that $A = (a_{ij})$ is an square matrix.

- A is said to be upper triangular if $a_{ij} = 0$ when $i > j$ i.e. when the entry is below the diagonal.
- A is said to be lower triangular if $a_{ij} = 0$ when $i < j$ i.e. when the entry is above the diagonal.

Triangular matrices, cont'd

Theorem (1.7.1)

- *The transpose of an upper triangular matrix is lower triangular and vice versa.*
- *The product of upper triangular matrices is upper triangular and the same for lower triangular matrices.*
- *A triangular matrix is invertible iff all its diagonal entries are non-zero.*
- *The inverse of an upper triangular matrix is upper triangular and the same for lower triangular matrices.*

Symmetric matrices

Definition

For a square matrix $A = (a_{ij})$ is said to be symmetric if $a_{ij} = a_{ji}$ for all i and j.

Theorem (1.7.2, 1.7.3 and 1.7.4)

Suppose that A and B are symmetric matrices. Then

- A^T *is symmetric;*
- $A + B$ *and* $A - B$ *are symmetric;*
- kA *is symmetric for all numbers* k*;*
- AB *is symmetric iff* $AB = BA$*; and*
- *If* A *is invertible then* A^{-1} *is also symmetric.*

Theorem (1.7.5)

If A *is invertible then* A^TA *and* AA^T *are invertible.*

Matrices as functions

- We would like to understand matrices as functions.
- The real question should then be: where are they functions from and where do they go to?

The vector space R^n

Definition

The set of all ordered n-tuples of real numbers will be denoted R^n; the elements of R^n are called vectors. Equivalently, we can think of R^n as the set of all $n \times 1$ matrices or column vectors i.e. matrices of the form

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}$$

The standard basis

Definition

The standard basis of R^n is the set of vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Linear combinations

Notice that every vector in R^n is a linear combination of $e_1, \ldots, e_n$.

The transformation T_A

Definition

Suppose that A is an $m \times n$ matrix then T_A is a function with domain R^n and range R^m, usually written

$$T_A : R^n \to R^m$$

defined by: for all $x \in R^n$, $T_A(x) = Ax$.

Theorem

If A is an $m \times n$ matrix, $x, y \in R^n$ and $\lambda \in R$ then

1. $T_A(x + y) = T_A(x) + T_A(y)$ *and*
2. $T_A(\lambda x) = \lambda T_A(x)$

Linear functions

Any function from R^n to R^m which the two properties from the theorem are called linear functions.

Linear functions

- Suppose that $T : R^n \to R^m$ is any linear function.
- Remember that if $x \in R^n$ then $x = \lambda_1 e_1 + \ldots + \lambda_n e_n$ for some $\lambda_1, \ldots, \lambda_n$.
- So $T(x) = \lambda_1 T(e_1) + \ldots + \lambda_n T(e_n)$.
- This says that every linear function is determined by its values on $e_1, \ldots, e_n$.
- Consider the matrix

$$A = (T(e_1)|T(e_2)|\ldots|T(e_n))$$

- We see that $T = T_A$.
- Conclusion: All linear functions from R^n to R^m are of the form T_A for some $m \times n$ matrix A.

Matrix multiplication and composition of functions

- The composition of two linear functions is a linear function.
- If A is $m \times k$ and B is $k \times n$ then we can form $T_A(T_B)$ - the composition of these two functions and it will be a linear function.
- By what was said on the previous slide, this linear function will be T_C for some C; what is C?
- $C = AB$.
- So matrix multiplication is what you get when you compose linear functions.