Invertible matrices

Definition

We say that a square matrix A is invertible if there is a square matrix B of the same size as A such that $AB = I$ and $BA = I$. We call B an inverse of A.

- If A is a square matrix and not invertible we say it is singular and sometimes we refer to invertible matrices as nonsingular.
- If a square matrix A has an inverse then that inverse is unique and we write A^{-1}. So A^{-1} is **the** inverse of A.

Properties of invertible matrices

Theorem (1.4.6)

Suppose that A and B are invertible $n \times n$ matrices. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem (1.4.7)

Suppose that A is an invertible matrix. Then

(a) *A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

(b) *For any natural number n, A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$.*

Properties of the transpose

Theorem (1.4.8 and 1.4.9)

For any matrices A and B for which the following make sense:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- *If A is invertible then* $(A^{-1})^T = (A^T)^{-1}$

Elementary matrices

Elementary row operations

The elementary row operations on a matrix are:

- Multiply a row by a non-zero constant
- Add a constant multiple of one row to another
- Interchange two rows

Definition

An elementary matrix is one obtained from an identity matrix by a single elementary row operation.

Main facts about elementary matrices

Theorem

If E is an elementary matrix and EA makes sense then if $EA = B$, B is the matrix obtained from A by applying the elementary row operation associated with E.

Corollary

All elementary matrices are invertible

An important theorem

Theorem (1.5.3)

The following are equivalent for a square matrix A:

1. *A is invertible.*
2. *The linear system $Ax = 0$ has only the trivial, 0, solution.*
3. *The reduced row echelon form of A is the identity matrix.*
4. *A is a product of elementary matrices.*