Reminders

- If S is any subset of a vector space V that spans V then there is a basis for V contained in S.
- To any $m \times n$ matrix A we associate a number of subspaces:
- The row space of A is the subspace of $\mathbb{R}^n$ spanned by the row vectors of A. Its dimension is called the rank of A; notice that the rank of A can be no bigger than the smaller of m and n.
- The nullspace of A is the subspace of $\mathbb{R}^n$ of all x such that $Ax = 0$. Its dimension is called the nullity of A.
- The dimension theorem guarantees that *rank*(A) + nullity(A) = n.

The column-space of a matrix

- We return to the problem of subspaces of $\mathbb{R}^n$. The question is: given vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$, how do we find a basis for the subspace W spanned by these vectors *from among these vectors.*
- Consider a matrix A formed by placing $v_1, \ldots, v_k$ in the columns; A is $n \times k$; let $B = rref(A)$, the reduced row-echelon form of A.
- The claim is that the vectors in A which correspond to the columns of B with leading 1's form a basis for W.
- Notice that this means that the dimension of W is the same as the rank of A.
- In general, for a matrix A which is $k \times n$, the subspace generated by the columns is called the column space of A, a subspace of $\mathbb{R}^k$, and its dimension is the same as the rank of A.

Another example

- $$A = \begin{pmatrix} 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 4 & -2 & -1 & 3 & -1 \\ 1 & 2 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$
- $$rref(A) = \begin{pmatrix} 1 & 2 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Another example, cont'd

- $$A^T = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ -1 & -2 & -1 & 0 \\ 0 & -1 & -1 & 1 \\ 1 & 3 & 2 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

- $$rref(A^T) = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Diagonalization and eigenspaces

- Suppose that A is an $n \times n$ matrix and it has λ as an eigenvalue.
- Remember that we refer to all $x \in \mathbb{R}^n$ such that $Ax = \lambda x$ as the eigenspace corresponding to λ for the matrix A; it is a subspace of $\mathbb{R}^n$.
- We say that A is diagonalizable if there is an invertible matrix P such that $P^{-1}AP = D$ with D a diagonal matrix. In fact, the numbers on the diagonal of D are the eigenvalues of A and the columns of P are eigenvectors.
- It follows that if A is diagonalizable then there is a basis of $\mathbb{R}^n$ made up of eigenvectors of A.
- Conversely, if there is a basis for $\mathbb{R}^n$ made up of eigenvectors of $\mathbb{R}^n$ then A is diagonalizable.

Diagonalization and eigenspaces, cont'd

- Buried in here, there is an algorithm for determining whether a matrix is diagonalizable or not.
- Suppose that λ is an eigenvalue for A. We call the multiplicity of λ in the characteristic polynomial its algebraic multiplicity. We call the dimension of the eignespace corresponding to λ it geometric multiplicity.

Theorem

A is diagonalizable iff for every eigenvalue λ of A, the algebraic multiplicity of λ equals the geometric multiplicity.

Where did the geometry go?

Definition

We call a basis S for $\mathbb{R}^n$ an orthogonal basis if for every distinct pair $u, v \in S$, $u \cdot v = 0$; we say that the basis is orthonormal if for every $u \in S$, $||u|| = 1$.

Fact

Any orthogonal set in $\mathbb{R}^n$ is linearly independent.

Coordinates with respect to an orthonormal basis

Suppose that $v_1, v_2, \ldots, v_n$ is an orthonormal basis for $\mathbb{R}^n$. If $v \in \mathbb{R}^n$ then

$$v = k_1 v_1 + \ldots k_n v_n$$

where $k_i = v \cdot v_i$ for $i = 1, \ldots, n$.

The Gram-Schmidt process

- Suppose we have a linearly independent set $u_1, \ldots, u_r$ which spans some subspace of $\mathbb{R}^n$. We would like to find an orthogonal basis for this same subspace.
- We construct an orthogonal set iteratively: let $v_1 = u_1$.
- Let $v_2 = u_2 - proj_{v_1}(u_2)$.
- Let $v_3 = u_3 - proj_{v_1}(u_3) - proj_{v_2}(u_3)$.
- In general, if we have already defined $v_1, \ldots, v_i$ then let

$$v_{i+1} = u_{i+1} - proj_{v_1}(u_{i+1}) - \ldots - proj_{v_i}(u_{i+1})$$

- Continue this process until you have dealt with all r vectors.
- The resulting $v_1, \ldots, v_r$ will be an orthogonal basis for W.