Basis

Definition

If V is a vector space and S is a subset of V then S is a basis for V if

1. S is linearly independent and
2. S spans V.

Theorem

If S is a basis for V then every vector $v \in V$ can be expressed in the form

$$v = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n$$

for distinct $v_1, \ldots, v_n \in S$ in exactly one way.

Dimension

Definition

We say that a vector space is finite-dimensional if it has a finite basis.

Theorem

If V is a vector space with a basis of n vectors then

1. *any subset of V with more than n vectors is linearly dependent, and*
2. *any subset of V with fewer than n vectors does not span V.*

Corollary

If V is a finite-dimensional vector space then all bases for V have the same size; we call this size the dimension of V.

The Plus/Minus Theorem

Theorem (Plus/Minus Theorem, 4.5.3)

Suppose that S is a non-empty subset of a vector space V. Then

1. *if S is linearly independent and $v \in V$ is not in span(S) then $S \cup \{v\}$ is linearly independent, and*
2. *if $v \in S$ can be expressed as a linear combination of vectors from $S \setminus \{v\}$ then span(S) = span$(S \setminus \{v\})$.*

Corollaries

Corollary

Suppose that V is an n-dimensional vector space and S is a subset of V.

1. *If S spans V then S contains a basis for V.*
2. *If S is linearly independent then S is contained in a basis for V.*
3. *If S contains exactly n vectors then S is a basis for V iff S is linearly independent.*

Corollary

If W is a subspace of a finite-dimensional vector space V then

1. *W is finite-dimensional;*
2. *$dim(W) \leq dim(V)$; and*
3. *$dim(W) = dim(V)$ iff $W = V$.*