Subspaces

Definition

Suppose that V is a vector space and W is a non-empty subset of V. We say that W is a subspace of V if with respect to the $+$ and scalar multiplication restricted to W from V, W is a vector space in its own right.

Theorem

If V is a vector space and W is a non-empty subset of V then W is a subspace iff W is closed under $+$ and scalar multiplication.

Span

Theorem

If V is a vector space and X is a subset of V then there is a subspace W containing X with the property that if any other subspace W' contains X then $W \subseteq W'$.

Definition

1. We call the W in the previous theorem the subspace spanned by X and write $\text{span}(X)$.
2. If $x_1, \ldots, x_n \in X$ and $k_1, \ldots, k_n \in \mathbb{R}$ then we call $k_1 x_1 + \ldots + k_n x_n$ a linear combination of elements of X.

Fact

If X is a subset of a vector space V then the set of all linear combinations of elements of X is the subspace spanned by X.

Dependence

Definition

If V is a vector space, $v \in V$ and $X \subseteq V$ then we say that v depends on X if v is in the span(X).

Fact

If X is a subset of a vector space V and $v \in V$ then v depends on X iff v can be expressed as a linear combination of vectors from X.

Linear independence

Definition

A subset S of a vector space V is linearly independent if whenever $v_1, \ldots, v_n \in S$ are distinct and

$$k_1 v_1 + \ldots + k_n v_n = 0$$

then it must be that

$$k_1 = \ldots = k_n = 0$$

S is said to be linearly dependent if it is not linearly independent.

Fact

If V is a vector space and $S \subseteq V$ then S is linearly dependent iff for some $v \in S$, v depends on $S \setminus \{v\}$ i.e. v can be written as a linear combination of vectors from $S \setminus \{v\}$.