Vector spaces

Definition

Suppose that V is a non-empty set, $+$ is function on pairs from V and for every $k \in \mathbb{R}$ and $u \in V$, ku is defined. We say that V together with these operations defines a vector space if the following axioms are satisfied for all $u, v, w \in V$ and $k, m \in \mathbb{R}$:

Vector spaces, cont'd

1. $u + v \in V$.
2. $u + v = v + u$
3. $u + (v + w) = (u + v) + w$
4. there is a $0 \in V$ such that $u + 0 = u$ for all $u \in V$
5. there is a $-u \in V$ such that $u + (-u) = 0$
6. $ku \in V$
7. $k(u + v) = ku + kv$
8. $(k + m)u = ku + mu$
9. $k(mu) = (km)u$
10. $1u = u$

Subspaces

Definition

Suppose that V is a vector space and W is a non-empty subset of V. We say that W is a subspace of V if with respect to the $+$ and scalar multiplication restricted to W from V, W is a vector space in its own right.

Theorem

If V is a vector space and W is a non-empty subset of V then W is a subspace iff W is closed under $+$ and scalar multiplication.

Span

Theorem

If V is a vector space and X is a subset of V then there is a subspace W containing X with the property that if any other subspace W' contains X then $W \subseteq W'$.

Definition

1. We call the W in the previous theorem the subspace spanned by X and write span(X).
2. If $x_1, \ldots, x_n \in X$ and $k_1, \ldots, k_n \in \mathbb{R}$ then we call $k_1 x_1 + \ldots + k_n x_n$ a linear combination of elements of X.

Fact

If X is a subset of a vector space V then the set of all linear combinations of elements of X is the subspace spanned by X.

Dependence

Definition

If V is a vector space, $v \in V$ and $X \subseteq V$ then we say that v depends on X if v is in the span(X).

Fact

If X is a subset of a vector space V and $v \in V$ then v depends on X iff v can be expressed as a linear combination of vectors from X.