Orthogonality

Definition

- We say that two vectors $u, v \in \mathbb{R}^n$ are orthogonal if $u \cdot v = 0$.
- We say that a set of vectors is orthogonal if any two distinct vectors in the set are orthogonal and the set is orthonormal if all the vectors have length 1.

Orthogonality, cont'd

Theorem (3.3.2)

If u and a are vectors in n-space and $a \neq 0$ then there are unique vectors w_1 and w_2 such that $u = w_1 + w_2$ with w_1 orthogonal to w_2 and w_1 a multiple of a. In fact,

$$w_1 = \frac{u \cdot a}{||a||^2} a \text{ and } w_2 = u - w_1$$

Theorem (Pythagoras)

In $\mathbb{R}^n$, if u and v are orthogonal then

$$||u||^2 + ||v||^2 = ||u + v||^2$$

Orthogonality and matrix multiplication

- Suppose that A is $m \times k$ with rows $r_1, \ldots, r_m$ and B is $k \times n$ with columns $c_1, \ldots, c_n$. Then the ij entry of AB is $r_i \cdot c_j$.
- In particular, if A is $m \times n$ and we know that x satisfies $Ax = 0$ then x is a vector in $\mathbb{R}^n$ which is orthogonal to all the row vectors.
- Geometrically then if x_0 satisfies $Ax = b$ then all solutions of $Ax = b$ can be obtained by adding x_0 to any vector which is orthogonal to all the row vectors of A i.e. a solution of $Ax = 0$.

Lines and planes in $\mathbb{R}^n$

Definition

- If x_0 and v are vectors in $\mathbb{R}^n$ then $x = x_0 + tv$ defines a line passing through x_0 parallel to v in $\mathbb{R}^n$.
- If x_0, u and v are vectors in $\mathbb{R}^n$ and u is not a multiple of v then $x = x_0 + t_1 u + t_2 v$ defines the plane in $\mathbb{R}^n$ passing through x_0 and parallel to u and v.

The cross product

Definition

Suppose that $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors in $\mathbb{R}^3$. We define the cross product of u with v, written $u \times v$, as

$$(u_2 v_3 - v_2 u_3, v_1 u_3 - u_1 v_3, u_1 v_2 - v_1 u_2)$$

- $u \times v$ can also be described geometrically as a vector which is both orthogonal to u and v, has length $||u|| ||v|| \sin(\theta)$ where θ is the angle between u and v, and is oriented according to the right-hand rule (watch my hands).
- Theorems 3.5.1 and 3.5.2 have many properties of the cross product which follow immediately from one or the other way of looking at the cross product.