## Definition

- $\mathbb{R}^n$ or real $n$-space is the collection of all $n$-tuples $v = (v_1, v_2, \ldots, v_n)$ where $v_1, \ldots, v_n \in \mathbb{R}$. We refer to elements of $\mathbb{R}^n$ as vectors.
- We define addition between vectors in $\mathbb{R}^n$ as follows:

$$(u_1, \ldots, u_n) + (v_1, \ldots, v_n) = (u_1 + v_1, \ldots, u_n + v_n)$$

- Scalar multiples of vectors in $\mathbb{R}^n$ is defined, for $r \in \mathbb{R}$ by

$$r(v_1, \ldots, v_n) = (rv_1, \ldots, rv_n)$$

**The norm on $\mathbb{R}^n$**

The norm of a vector $v = (v_1, \ldots, v_n)$, written $||v||$, is $\sqrt{v_1^2 + \ldots + v_n^2}$ and there is a corresponding notion of distance $d(u, v) = ||u - v||$.

### The norm on $\mathbb{R}^n$

The norm of a vector $v = (v_1, \ldots, v_n)$, written $||v||$, is $\sqrt{v_1^2 + \ldots + v_n^2}$ and there is a corresponding notion of distance $d(u, v) = ||u - v||$.

### Theorem (3.2.1)

*For $v \in \mathbb{R}^n$,*

**The norm on $\mathbb{R}^n$**

The norm of a vector $v = (v_1, \ldots, v_n)$, written $||v||$, is $\sqrt{v_1^2 + \ldots + v_n^2}$ and there is a corresponding notion of distance $d(u, v) = ||u - v||$.

**Theorem (3.2.1)**

*For $v \in \mathbb{R}^n$,*

- $||v|| \geq 0$

## The norm on $\mathbb{R}^n$

The norm of a vector $v = (v_1, \ldots, v_n)$, written $||v||$, is $\sqrt{v_1^2 + \ldots + v_n^2}$ and there is a corresponding notion of distance $d(u, v) = ||u - v||$.

## Theorem (3.2.1)

*For $v \in \mathbb{R}^n$,*

- $||v|| \geq 0$
- $||v|| = 0$ *iff* $v = 0$

**The norm on $\mathbb{R}^n$**

The norm of a vector $v = (v_1, \ldots, v_n)$, written $||v||$, is $\sqrt{v_1^2 + \ldots + v_n^2}$ and there is a corresponding notion of distance $d(u, v) = ||u - v||$.

**Theorem (3.2.1)**

*For $v \in \mathbb{R}^n$,*

- $||v|| \geq 0$
- $||v|| = 0$ *iff* $v = 0$
- $||kv|| = |k|||v||$

# The geometry of $\mathbb{R}^n$

## Definition

For two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ we define the dot product

$$(u_1, \ldots, u_n) \cdot (v_1, \ldots, v_n) = u_1 v_1 + \ldots + u_n v_n$$

and the angle $\theta$ between non-zero vectors $u$ and $v$ by

$$\cos(\theta) = \frac{u \cdot v}{||u||||v||}$$

# The geometry of $\mathbb{R}^n$, cont'd

- Theorems 3.2.2 and 3.2.3 have many valuable parts; I draw your attention to the part which says: for a vector $v$, $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$.

# The geometry of $\mathbb{R}^n$, cont'd

- Theorems 3.2.2 and 3.2.3 have many valuable parts; I draw your attention to the part which says: for a vector $v$, $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$.
- Theorem 3.2.4, the Cauchy-Schwartz inequality says that for all vectors $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq ||u||||v||$.

# The geometry of $\mathbb{R}^n$, cont'd

- Theorems 3.2.2 and 3.2.3 have many valuable parts; I draw your attention to the part which says: for a vector $v$, $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$.
- Theorem 3.2.4, the Cauchy-Schwartz inequality says that for all vectors $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq ||u||||v||$.
- This allows us to prove the triangle inequality: for all vectors $u, v \in \mathbb{R}^n$, $||u + v|| \leq ||u|| + ||v||$.

# The geometry of $\mathbb{R}^n$, cont'd

- Theorems 3.2.2 and 3.2.3 have many valuable parts; I draw your attention to the part which says: for a vector $v$, $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$.
- Theorem 3.2.4, the Cauchy-Schwartz inequality says that for all vectors $u, v \in \mathbb{R}^n$, $|u \cdot v| \leq ||u||||v||$.
- This allows us to prove the triangle inequality: for all vectors $u, v \in \mathbb{R}^n$, $||u + v|| \leq ||u|| + ||v||$.
- Theorems 3.2.6 and 3.2.7 express in $n$-space two geometric results: for all vectors $u, v \in \mathbb{R}^n$,

$$||u + v||^2 + ||u - v||^2 = 2(||u||^2 + ||v||^2) \text{ and}$$

$$u \cdot v = \frac{1}{4}||u + v||^2 - \frac{1}{4}||u - v||^2$$

# Orthogonality

### Definition

# Orthogonality

## Definition

- We say that two vectors $u, v \in \mathbb{R}^n$ are orthogonal if $u \cdot v = 0$.

# Orthogonality

## Definition

- We say that two vectors $u, v \in \mathbb{R}^n$ are orthogonal if $u \cdot v = 0$.
- We say that a set of vectors is orthogonal if any two distinct vectors in the set are orthogonal and the set is orthonormal if all the vectors have length 1.

# Orthogonality, cont'd

**Theorem (3.3.2)**

*If $u$ and $a$ are vectors in n-space and $a \neq 0$ then there are unique vectors $w_1$ and $w_2$ such that $u = w_1 + w_2$ with $w_1$ orthogonal to $w_2$ and $w_1$ a multiple of $a$. In fact,*

$$w_1 = \frac{u \cdot a}{||a||^2} a \text{ and } w_2 = u - w_1$$

# Orthogonality, cont'd

**Theorem (3.3.2)**

*If $u$ and $a$ are vectors in n-space and $a \neq 0$ then there are unique vectors $w_1$ and $w_2$ such that $u = w_1 + w_2$ with $w_1$ orthogonal to $w_2$ and $w_1$ a multiple of $a$. In fact,*

$$w_1 = \frac{u \cdot a}{||a||^2} a \text{ and } w_2 = u - w_1$$

**Theorem (Pythagoras)**

*In $\mathbb{R}^n$, if $u$ and $v$ are orthogonal then*

$$||u||^2 + ||v||^2 = ||u + v||^2$$