Augmented matrices

For the linear system

$$\begin{array}{ccccc} a_{11}x_1 + a_{12}x_2 + & \dots & +a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + & \dots & +a_{2n}x_n & = & b_2 \\ \vdots & \ddots & \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + & \dots & +a_{mn}x_n & = & b_m \end{array}$$

the augmented matrix is

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

Elementary row operations

To solve a linear system one performs a series of algebraic operations:

- Multiply an equation by a non-zero constant
- Add a constant multiple of one equation to another
- Interchange equations

The main fact is that doing these operations doesn't change the set of solutions of the linear system.

The corresponding *elementary row operations* on an augmented matrix are:

- Multiply a row by a non-zero constant
- Add a constant multiple of one row to another
- Interchange two rows

Echelon forms

A matrix is in **row echelon form** if

1. If a row isn't entirely zeroes then the left most non-zero entry is a 1; we call this 1 the row's leading 1.
2. All zero rows are grouped contiguously at the bottom of the matrix.
3. In any two consecutive rows which are both non-zero, the leading 1 of the upper row is to the left of the leading 1 of the lower row.
4. If additionally, any column which contains a leading 1 has zeroes elsewhere then we say the matrix is in **reduced row echelon form**.

Gaussian and Gauss-Jordan elimination

1. Given the augmented matrix for a system of linear equations, find the left most column which is not all zero.
2. Interchange rows so that the non-zero entry from the previous step is the top row.
3. If the non-zero entry you have found is a then divide the top row by a so that its left most entry is 1.
4. Work down row by row adding multiples of the first row to each row to guarantee that all entries below this 1 are zero.
5. Now ignore the top row and repeat the process with the rows you have remaining until there are no more rows left; this gets the matrix into row echelon form.
6. For reduced row echelon form, start with the right most leading 1 and working left, add suitable multiples of that row to the rows above to guarantee that all entries above leading 1's are 0.

Homogeneous systems

- We say that a system of linear equations is a homogeneous system if all the constants in the equations (the right-hand sides) are 0.
- Any homogeneous system of linear equations has at least one solution.
- Theorem: Any homogeneous system with more unknowns than equations has infinitely many solutions.