The complex numbers

- Introduce a new quantity, i, such that $i^2 = -1$.
- The complex numbers are then all expressions of the form $a + bi$ where a and b are real numbers.

Operations on the complex numbers

- Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- Multiplication:

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

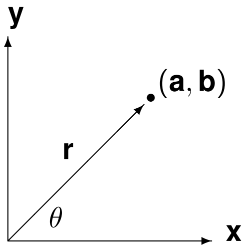
Multiplicative inverse

Every non-zero complex number has a multiplicative inverse. That is, if z_1 is not zero then the equation, in the unknown z, $z_1 z = 1$ has a solution.

The conjugate

- If $z = a + bi$ then $\bar{z}$, the conjugate of z, is $a - bi$.
- Notice that $z\bar{z} = a^2 + b^2$ so
- $$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$$

The complex plane



- $r = \sqrt{a^2 + b^2}$; this is called the modulus of the complex number $z = a + bi$ and written $|z|$.
- We saw that $z \cdot \bar{z} = |z|^2$.
- θ is an argument for $a + bi$ and is only determined up to multiples of 2π.
- $a = r\cos(\theta)$ and $b = r\sin(\theta)$ so $z = r(\cos(\theta) + i\sin(\theta))$.

Multiplication and division in polar form

- If

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \text{ and } z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$$

- then $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ and
- if $z_2 \neq 0$ then $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$.
- A special case occurs when you compute z^n where $z = r(\cos(\theta) + i\sin(\theta))$:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

- We can also compute n^{th} roots of $r(\cos(\theta) + i\sin(\theta))$; they are

$$r^{1/n}(\cos(\frac{\theta}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\theta}{n} + \frac{2k\pi}{n})) \text{ for } k = 0, 1, \ldots, n-1$$

A beautiful formula and a caveat going forward

- Exponentiation can be defined on complex numbers as follows:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

- Consider then that if $\theta = \pi$ then

$$e^{i\pi} + 1 = 0$$

Linear algebra over the complex numbers

Everything we have done in the course to date - linear systems, matrices, determinants, eigenvalues, diagonalization - all goes through **unchanged** if we are using complex numbers instead of real numbers. Going forward it will be considered fair to have complex entries in matrices and to consider complex roots of polynomials.