Our probabilistic model

- We have an initial distribution of objects into n states; we will write $\mathbf{x}(0) = (x_1(0), x_2(0), \ldots, x_n(0))$ for the initial distribution; $x(i)$ is the number or proportion of objects in state i.
- At each time step, we assume that an object transitions from state i to state j with probability q_{ji}.
- If $\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))$ represents the distribution after k steps then we have

$$x_j(k+1) = q_{j1}x_1(k) + q_{j2}x_2(k) + \ldots + q_{jn}x_n(k)$$

for all j and k.

Our probabilistic model, cont'd

- So if Q is the matrix

$$\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

  then we have

$$\mathbf{x}(k+1) = Q\mathbf{x}(k) \text{ and } \mathbf{x}(k) = Q^k\mathbf{x}(0)$$

- A column vector with non-negative entries that sum to 1 is called a probability vector and a matrix whose columns are probability vectors is called a stochastic matrix.
- Notice that our assumption is that Q is a stochastic matrix and the transition model that we have described is call a Markov chain.

Stochastic matrices

- **Fact:** 1 is an eigenvalue for any stochastic matrix.
- We say that a system is in equilibrium if $Qx = x$ i.e. if x is an eigenvector for the eigenvalue 1for Q.
- Under very mild assumptions on a stochastic matrix Q, Q will have n distinct eigenvalues and all but 1 will have absolute value less than 1.
- Under these weak assumptions, if one starts with any probability vector $\mathbf{x}$ and $\mathbf{v}$ is the probability vector corresponding to the eigenvalue 1 then $\lim_{n\to\infty} Q^n\mathbf{x} = \mathbf{v}$.

The complex numbers

- Introduce a new quantity, i, such that $i^2 = -1$.
- The complex numbers are then all expressions of the form $a + bi$ where a and b are real numbers.

Operations on the complex numbers

- Addition:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- Multiplication:

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

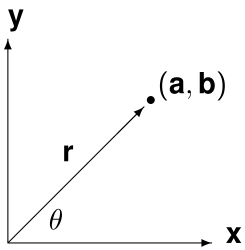
Multiplicative inverse

Every non-zero complex number has a multiplicative inverse. That is, if z_1 is not zero then the equation, in the unknown z, $z_1 z = 1$ has a solution.

The conjugate

- If $z = a + bi$ then $\bar{z}$, the conjugate of z, is $a - bi$.
- Notice that $z\bar{z} = a^2 + b^2$ so
- $$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$$

The complex plane



- $r = \sqrt{a^2 + b^2}$; this is called the modulus of the complex number $z = a + bi$ and written $|z|$.
- We saw that $z \cdot \bar{z} = |z|^2$.
- θ is an argument for $a + bi$ and is only determined up to multiples of 2π.
- $a = r\cos(\theta)$ and $b = r\sin(\theta)$ so $z = r(\cos(\theta) + i\sin(\theta))$.