Diagonalizability: a special case

Diagonalizability

- We say that a matrix is diagonalizable if it is similar to a diagonal matrix.
- If A is $n \times n$ and the characteristic equation of A has n distinct roots then A is diagonalizable.

An algorithm for diagonalizing: a special case

If A is $n \times n$ and has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ then to find a diagonal matrix similar to A, do the following:

- for each $i = 1, \ldots n$, find a non-zero column vector v_i which is an eigenvector for λ_i i.e. find v_i such that $Av_i = \lambda_i v_i$.
- Let P be the matrix formed by placing v_i in the i^{th} column.
- Then P is invertible and $P^{-1}AP = D$ where D is the diagonal matrix with entry λ_i in the i^{th} place on the diagonal.

Our probabilistic model

- We have an initial distribution of objects into n states; we will write $\mathbf{x}(0) = (x_1(0), x_2(0), \ldots, x_n(0))$ for the initial distribution; $x(i)$ is the number or proportion of objects in state i.
- At each time step, we assume that an object transitions from state i to state j with probability q_{ji}.
- If $\mathbf{x}(k) = (x_1(k), x_2(k), \ldots, x_n(k))$ represents the distribution after k steps then we have

$$x_j(k+1) = q_{j1}x_1(k) + q_{j2}x_2(k) + \ldots + q_{jn}x_n(k)$$

for all j and k.

Our probabilistic model, cont'd

- So if Q is the matrix

$$\begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$$

  then we have

$$\mathbf{x}(k+1) = Q\mathbf{x}(k) \text{ and } \mathbf{x}(k) = Q^k\mathbf{x}(0)$$

- A column vector with non-negative entries that sum to 1 is called a probability vector and a matrix whose columns are probability vectors is called a stochastic matrix.
- Notice that our assumption is that Q is a stochastic matrix and the transition model that we have described is call a Markov chain.

Stochastic matrices

- **Fact:** 1 is an eigenvalue for any stochastic matrix.
- We say that a system is in equilibrium if $Qx = x$ i.e. if x is an eigenvector for the eigenvalue 1for Q.
- Under very mild assumptions on a stochastic matrix Q, Q will have n distinct eigenvalues and all but 1 will have absolute value less than 1.
- Under these weak assumptions, if one starts with any probability vector $\mathbf{x}$ and $\mathbf{v}$ is the probability vector corresponding to the eigenvalue 1 then $\lim_{n\to\infty} Q^n\mathbf{x} = \mathbf{v}$.