Definition

For an $n \times n$ matrix A, we define

- The *ij* minor of A, M_{ij}, is the determinant of the square matrix obtained from A by deleting the i^{th} row and j^{th} column of A.
- The *ij* cofactor of A is $(-1)^{i+j}M_{ij}$.
- The cofactor expansion along the i^{th} row is

$$a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

- The cofactor expansion along the j^{th} column is

$$a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

Some easy facts

Theorem

Any cofactor expansion of a square matrix A, along any row or any column, always yields the same number and we call that number the determinant of A.

- If A is a triangular matrix then $det(A)$ is the product of the diagonal entries.
- If a square matrix has a row or column which is entirely zero then its determinant is 0.
- If A is a square matrix then $det(A) = det(A^T)$.
- If B is a square matrix obtained by multiplying a row or column of A by k then $det(B) = kdet(A)$.
- If B is obtained from A by exchanging two rows then $det(B) = -det(A)$.

Slightly more work

The effect of adding a multiple of one row to another

If a matrix B is obtained from A by adding a multiple of one row to another then $det(B) = det(A)$.

An efficient algorithm for finding the determinant

Start with a square matrix A and row reduce it to a triangular matrix B using only row changes and adding multiples of one row to another. Then $det(A)$ will be $det(B)$ multiplied by $(-1)^N$ where N is the number of row changes you did and $det(B)$ can be determined by multiplying the diagonal elements of B.

Determinants of elementary matrices

Corollary (Theorem 2.2.4)

- *If E is an elementary matrix obtained from I by multiplying a row by k then $\det(E) = k$.*
- *If E is an elementary matrix obtained from I by interchanging two rows then $\det(E) = -1$.*
- *If E is an elementary matrix obtained from I by adding a multiple of one row to another then $\det(E) = 1$.*