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10.1 COMPLEX NUMBERS

In this section we shall review the definition of a complex number and discuss the addition, subtraction, and multiplication of such numbers. We will also consider matrices with complex entries and explain how addition and subtraction of complex numbers can be viewed as operations on vectors.

Complex Numbers

Since $x^2 \geq 0$ for every real number *x*, the equation $x^2 = -1$ has no real solutions. To deal with this problem, mathematicians of the eighteenth century introduced the "imaginary" number,

$$i = \sqrt{-1}$$

which they assumed had the property

$$i^2 = (\sqrt{-1})^2 = -1$$

but which otherwise could be treated like an ordinary number. Expressions of the form

$$a + bi \tag{1}$$

where *a* and *b* are real numbers, were called "complex numbers," and these were manipulated according to the standard rules of arithmetic with the added property that $i^2 = -1$.

By the beginning of the nineteenth century it was recognized that a complex number 1 could be regarded as an alternative symbol for the ordered pair

$$(a, b)$$

of real numbers, and that operations of addition, subtraction, multiplication, and division could be defined on these ordered pairs so that the familiar laws of arithmetic hold and $i^2 = -1$. This is the approach we will follow.

DEFINITION

A ***complex number*** is an ordered pair of real numbers, denoted either by (a, b) or by $a + bi$, where $i^2 = -1$.

EXAMPLE 1 Two Notations for a Complex Number

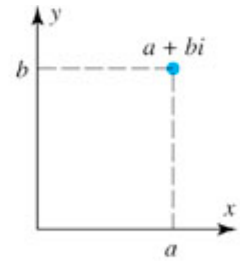
Some examples of complex numbers in both notations are as follows:

Ordered Pair	**Equivalent Notation**
(3, 4)	$3+4i$
(−1, 2)	$-1+2i$
(0, 1)	$0+i$
(2, 0)	$2+0i$
(4, −2)	$4+(-2)i$

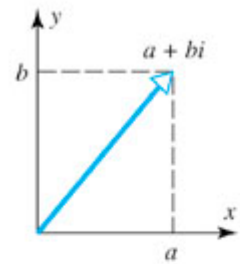
For simplicity, the last three complex numbers would usually be abbreviated as

$$0+i=i, \qquad 2+0i=2, \qquad 4+(-2)i=4-2i$$

Geometrically, a complex number can be viewed as either a point or a vector in the xy-plane (Figure 10.1.1).



(*a*) Complex number as a point



(*b*) Complex number as a vector

Figure 10.1.1

EXAMPLE 2 Complex Numbers as Points and as Vectors

Some complex numbers are shown as points in Figure 10.1.2*a* and as vectors in Figure 10.1.2*b*.

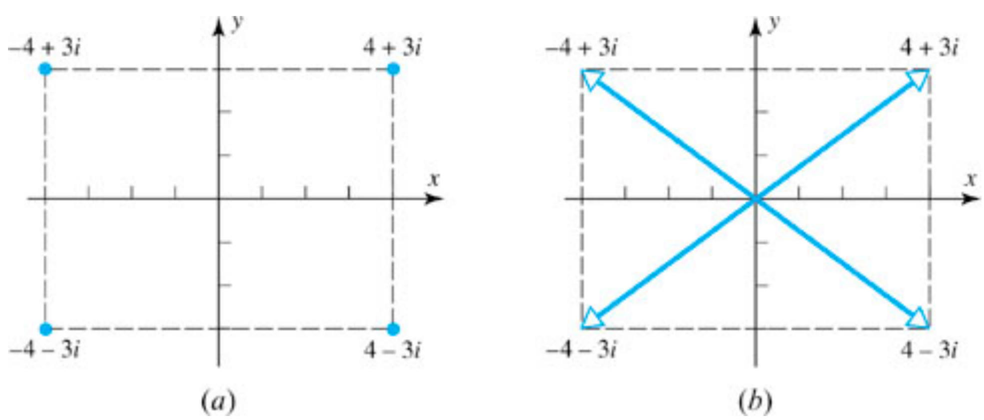


Figure 10.1.2

The Complex Plane

Sometimes it is convenient to use a single letter, such as z, to denote a complex number. Thus we might write

$$z = a + bi$$

The real number a is called the ***real part of z***, and the real number b is called the ***imaginary part of z***. These numbers are denoted by $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$, respectively. Thus

$$\mathrm{Re}\,(4-3i) = 4 \qquad \text{and} \qquad \mathrm{Im}\,(4-3i) = -3$$

When complex numbers are represented geometrically in an xy-coordinate system, the x-axis is called the ***real axis***, the y-axis is called the ***imaginary axis***, and the plane is called the ***complex plane*** (Figure 10.1.3). The resulting plot is called an ***Argand diagram***.

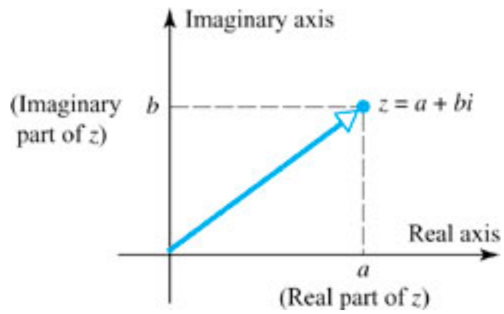


Figure 10.1.3
Argand diagram.

Operations on Complex Numbers

Just as two vectors in R^2 are defined to be equal if they have the same components, so we define two complex numbers to be equal if their real parts are equal and their imaginary parts are equal:

DEFINITION

Two complex numbers, $a + bi$ and $c + di$, are defined to be ***equal***, written

$$a + bi = c + di$$

if $a = c$ and $b = d$.

If $b = 0$, then the complex number $a + bi$ reduces to $a + 0i$, which we write simply as a. Thus, for any real number a,

$$a = a + 0i$$

so the real numbers can be regarded as complex numbers with an imaginary part of zero. Geometrically, the real numbers correspond to points on the real axis. If we have $a = 0$, then $a + bi$

reduces to $0 + bi$, which we usually write as bi. These complex numbers, which correspond to points on the imaginary axis, are called ***pure imaginary numbers***.

Just as vectors in R^2 are added by adding corresponding components, so complex numbers are added by adding their real parts and adding their imaginary parts:

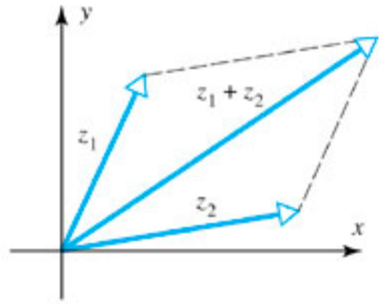
$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{2}$$

The operations of subtraction and multiplication by a *real* number are also similar to the corresponding vector operations in R^2:

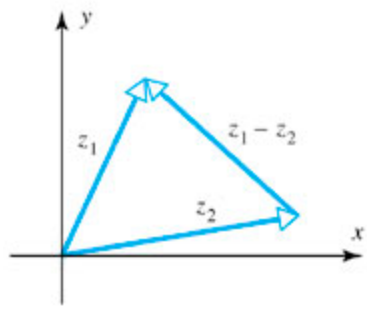
$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{3}$$

$$k(a + bi) = (ka) + (kb)i, \qquad k \text{ real} \tag{4}$$

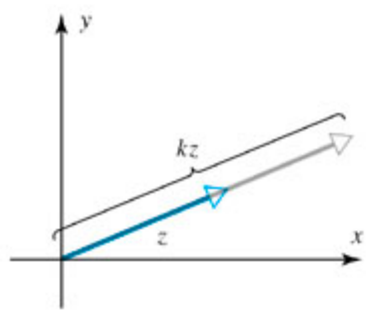
Because the operations of addition, subtraction, and multiplication of a complex number by a real number parallel the corresponding operations for vectors in R^2, the familiar geometric interpretations of these operations hold for complex numbers (see Figure 10.1.4).



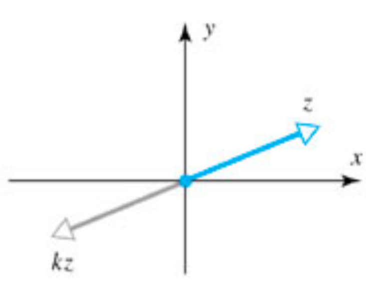
(a) The sum of two complex numbers



(b) The difference of two complex numbers



(c) The product of a complex number z and a positive real number k



(d) The product of a complex number z and a negative

Figure 10.1.4

It follows from 4 that $(-1)z + z = 0$ (verify), so we denote $(-1)z$ as $-z$ and call it the ***negative of z***.

EXAMPLE 3 Adding, Subtracting, and Multiplying by Real Numbers

If $z_1 = 4 - 5i$ and $z_2 = -1 + 6i$, find $z_1 + z_2$, $z_1 - z_2$, $3z_1$, and $-z_2$.

Solution

$$\begin{aligned} z_1 + z_2 &= (4 - 5i) + (-1 + 6i) = (4 - 1) + (-5 + 6)i = 3 + i \\ z_1 - z_2 &= (4 - 5i) - (-1 + 6i) = (4 + 1) + (-5 - 6)i = 5 - 11i \\ 3z_1 &= 3(4 - 5i) = 12 - 15i \\ -z_2 &= (-1)z_2 = (-1)(-1 + 6i) = 1 - 6i \end{aligned}$$

So far, there has been a parallel between complex numbers and vectors in R^2. However, we now define multiplication of complex numbers, an operation with no vector analog in R^2. To motivate the definition, we expand the product

$$(a + bi)(c + di)$$

following the usual rules of algebra but treating i^2 as −1. This yields

$$\begin{aligned} (a + bi)(c + di) &= ac + bdi^2 + adi + bci \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

which suggests the following definition:

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i \qquad (5)$$

EXAMPLE 4 Multiplying Complex Numbers

$$(3+2i)(4+5i) = (3\cdot 4 - 2\cdot 5) + (3\cdot 5 + 2\cdot 4)i$$
$$= 2 + 23i$$
$$(4-i)(2-3i) = [4\cdot 2 - (-1)(-3)] + [(4)(-3) + (-1)(2)]i$$
$$= 5 - 14i$$
$$i^2 = (0+i)(0+i) = (0\cdot 0 - 1\cdot 1) + (0\cdot 1 + 1\cdot 0)i = -1$$

We leave it as an exercise to verify the following rules of complex arithmetic:

$$z_1 + z_2 = z_2 + z_1$$
$$z_1 z_2 = z_2 z_1$$
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$
$$z_1(z_2 z_3) = (z_1 z_2) z_3$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$
$$0 + z = z$$
$$z + (-z) = 0$$
$$1 \cdot z = z$$

These rules make it possible to multiply complex numbers without using Formula 5 directly. Following the procedure used to motivate this formula, we can simply multiply each term of $a + bi$ by each term of $c + di$, set $i^2 = -1$, and simplify.

EXAMPLE 5 Multiplication of Complex Numbers

$$(3+2i)(4+i) = 12 + 3i + 8i + 2i^2 = 12 + 11i - 2 = 10 + 11i$$
$$\left(5 - \frac{1}{2}i\right)(2+3i) = 10 + 15i - i - \frac{3}{2}i^2 = 10 + 14i + \frac{3}{2} = \frac{23}{2} + 14i$$
$$i(1+i)(1-2i) = i(1 - 2i + i - 2i^2) = i(3-i) = 3i - i^2 = 1 + 3i$$

Remark Unlike the real numbers, there is no size ordering for the complex numbers. Thus, the order symbols <, ≤, >, and ≥ are not used with complex numbers.

Now that we have defined addition, subtraction, and multiplication of complex numbers, it is possible to add, subtract, and multiply matrices with complex entries and to multiply a matrix by a complex number. Without going into detail, we note that the matrix operations and terminology discussed in

Chapter 1 carry over without change to matrices with complex entries.

EXAMPLE 6 Matrices with Complex Entries

If

$$A = \begin{bmatrix} 1 & -i \\ 1+i & 4-i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix}$$

then

$$A+B = \begin{bmatrix} 1+i & 1-2i \\ 3-2i & 8-i \end{bmatrix}, \qquad A-B = \begin{bmatrix} 1-i & -1 \\ -1+4i & -i \end{bmatrix}$$

$$iA = \begin{bmatrix} i & -i^2 \\ i+i^2 & 4i-i^2 \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1+i & 1+4i \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & -i \\ 1+i & 4-i \end{bmatrix} \begin{bmatrix} i & 1-i \\ 2-3i & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot i + (-i) \cdot (2-3i) & 1 \cdot (1-i) + (-i) \cdot 4 \\ (1+i) \cdot i + (4-i) \cdot (2-3i) & (1+i) \cdot (1-i) + (4-i) \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} -3-i & 1-5i \\ 4-13i & 18-4i \end{bmatrix} \end{aligned}$$

Exercise Set 10.1

[Click here for Just Ask!](#)

1. In each part, plot the point and sketch the vector that corresponds to the given complex number.

(a) $2+3i$

(b) -4

(c) $-3-2i$

(d) $-5i$

2. Express each complex number in Exercise 1 as an ordered pair of real numbers.

3. In each part, use the given information to find the real numbers *x* and *y*.

(a) $x - iy = -2 + 3i$

(b) $(x+y)+(x-y)i = 3+i$

4. Given that $z_1 = 1-2i$ and $z_2 = 4+5i$, find

(a) $z_1 + z_2$

(b) $z_1 - z_2$

(c) $4z_1$

(d) $-z_2$

(e) $3z_1 + 4z_2$

(f) $\frac{1}{2}z_1 - \frac{3}{2}z_2$

5. In each part, solve for *z*.

(a) $z + (1 - i) = 3 + 2i$

(b) $-5z = 5 + 10i$

(c) $(i - z) + (2z - 3i) = -2 + 7i$

6. In each part, sketch the vectors z_1, z_2, $z_1 + z_2$, and $z_1 - z_2$.

(a) $z_1 = 3 + i, z_2 = 1 + 4i$

(b) $z_1 = -2 + 2i, z_2 = 4 + 5i$

7. In each part, sketch the vectors z and kz.

(a) $z = 1 + i, k = 2$

(b) $z = -3 - 4i, k = -2$

(c) $z = 4 + 6i, k = \frac{1}{2}$

8. In each part, find real numbers k_1 and k_2 that satisfy the equation.

(a) $k_1 i + k_2(1 + i) = 3 - 2i$

(b) $k_1(2+3i)+k_2(1-4i)=7+5i$

9. In each part, find z_1z_2, z_1^2, and z_2^2.

(a) $z_1=3i, z_2=1-i$

(b) $z_1=4+6i, z_2=2-3i$

(c) $z_1=\frac{1}{3}(2+4i), z_2=\frac{1}{2}(1-5i)$

10. Given that $z_1=2-5i$ and $z_2=-1-i$, find

(a) $z_1-z_1z_2$

(b) $(z_1+3z_2)^2$

(c) $[z_1+(1+z_2)]^2$

(d) $iz_2-z_1^2$

In Exercises 11–18 perform the calculations and express the result in the form $a+bi$.

11. $(1+2i)(4-6i)^2$

12. $(2-i)(3+i)(4-2i)$

13. $(1-3i)^3$

14. $i(1+7i)-3i(4+2i)$

15. $\left[(2+i)\left(\frac{1}{2}+\frac{3}{4}i\right)\right]^2$

16. $(\sqrt{2}+i)-i\sqrt{2}(1+\sqrt{2}i)$

17. $(1+i+i^2+i^3)^{100}$

18. $(3-2i)^2-(3+2i)^2$

19. Let

$$A=\begin{bmatrix}1 & i\\ -i & 3\end{bmatrix}, \qquad B=\begin{bmatrix}2 & 2+i\\ 3-i & 4\end{bmatrix}$$

Find

(a) $A+3i\,B$

(b) BA

(c) AB

(d) B^2-A^2

20. Let

$$A=\begin{bmatrix}3+2i & 0\\ -i & 2\\ 1+i & 1-i\end{bmatrix}, \qquad B=\begin{bmatrix}-i & 2\\ 0 & i\end{bmatrix}, \qquad C=\begin{bmatrix}-1-i & 0 & -i\\ 3 & 2i & -5\end{bmatrix}$$

Find

(a) $A(BC)$

(b) $(BC)A$

(c) $(CA)B^2$

(d) $(1+i)(AB)+(3-4i)A$

21. Show that

(a) $\operatorname{Im}(iz)=\operatorname{Re}(z)$

(b) $\operatorname{Re}(iz)=-\operatorname{Im}(z)$

22. In each part, solve the equation by the quadratic formula and check your results by substituting the solutions into the given equation.

(a) $z^2+2z+2=0$

(b) $z^2-z+1=0$

23.

(a) Show that if n is a positive integer, then the only possible values for i^n are 1, −1, i, and $-i$.

(b) Find i^{2509}.

24. Prove: If $z_1 z_2 = 0$, then $z_1 = 0$ or $z_2 = 0$.

25. Use the result of Exercise 24 to prove: If $zz_1 = zz_2$ and $z \neq 0$, then $z_1 = z_2$.

26. Prove that for all complex numbers z_1, z_2, and z_3,

(a) $z_1 + z_2 = z_2 + z_1$

(b) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

27. Prove that for all complex numbers z_1, z_2, and z_3,

(a) $z_1 z_2 = z_2 z_1$

(b) $z_1(z_2 z_3) = (z_1 z_2) z_3$

28. Prove that $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all complex numbers z_1, z_2, and z_3.

29. In quantum mechanics the ***Dirac matrices*** are

$$\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \alpha_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

(a) Prove that $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = I$.

30. Describe the set of all complex numbers $z = a + bi$ such that $a^2 + b^2 = 1$. Show that if z_1, z_2 are such numbers, then so is $z_1 z_2$.

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10.2
DIVISION OF COMPLEX NUMBERS

In the last section we defined multiplication of complex numbers. In this section we shall define division of complex numbers as the inverse of multiplication.

We begin with some preliminary ideas.

Complex Conjugates

If $z = a + bi$ is any complex number, then the ***complex conjugate*** of z (also called the ***conjugate*** of z) is denoted by the symbol $\bar{z}$ (read " z bar" or " z conjugate") and is defined by

$$\bar{z} = a - bi$$

In words, $\bar{z}$ is obtained by reversing the sign of the imaginary part of z. Geometrically, $\bar{z}$ is the reflection of z about the real axis (Figure 10.2.1).

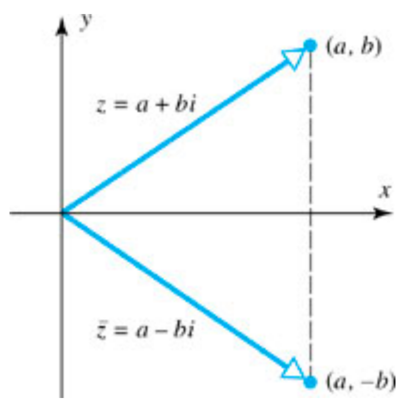


Figure 10.2.1
The conjugate of a complex number.

EXAMPLE 1 Examples of Conjugates

$$\begin{array}{ll} z = 3 + 2i & \bar{z} = 3 - 2i \\ z = -4 - 2i & \bar{z} = -4 + 2i \\ z = i & \bar{z} = -i \\ z = 4 & \bar{z} = 4 \end{array}$$

Remark The last line in Example 1 illustrates the fact that a real number is the same as its conjugate. More precisely, it can be shown (Exercise 22) that $z = \bar{z}$ if and only if z is a real number.

If a complex number z is viewed as a vector in R^2, then the norm or length of the vector is called the modulus of z. More precisely:

DEFINITION

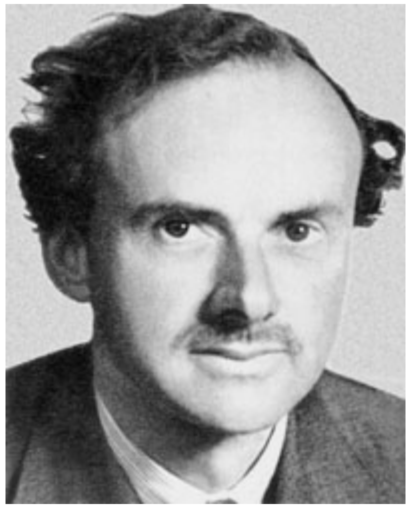
The ***modulus*** of a complex number $z = a + bi$, denoted by $|z|$, is defined by

$$|z| = \sqrt{a^2 + b^2} \tag{1}$$

If $b = 0$, then $z = a$ is a real number, and

$$|z| = \sqrt{a^2 + 0^2} = \sqrt{a^2} = |a|$$

so the modulus of a real number is simply its absolute value. Thus the modulus of z is also called the ***absolute value*** of z.



Paul Adrien Maurice Dirac *(1902–1984)* was a British theoretical physicist who devised a new form of quantum mechanics and a theory that predicted electron spin and the existence of a fundamental atomic particle called a positron. He received the Nobel Prize for physics in 1933 and the medal of the Royal Society in 1939.

EXAMPLE 2 Modulus of a Complex Number

Find $|z|$ if $z = 3 - 4i$.

Solution

From 1, with $a = 3$ and $b = -4$, $|z| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$.

The following theorem establishes a basic relationship between $\bar{z}$ and $|z|$.

THEOREM 10.2.1

For any complex number z,

$$z\bar{z} = |z|^2$$

Proof If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - abi + bai - b^2 i^2 = a^2 + b^2 = |z|^2$$

■

Division of Complex Numbers

We now turn to the division of complex numbers. Our objective is to define division as the inverse of multiplication. Thus, if $z_2 \neq 0$, then our definition of $z = z_1 / z_2$ should be such that

$$z_1 = z_2 z \tag{2}$$

Our procedure will be to prove that 2 has a unique solution for z if $z_2 \neq 0$, and then to define z_1 / z_2 to be this value of z. As with real numbers, division by zero is not allowed.

THEOREM 10.2.2

If $z_2 \neq 0$, then Equation 2 has a unique solution, which is

$$z = \frac{1}{|z_2|^2} z_1 \bar{z}_2 \tag{3}$$

Proof Let $z = x + iy$, $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$. Then 2 can be written as

$$x_1 + iy_1 = (x_2 + iy_2)(x + iy)$$

or

$$x_1 + iy_1 = (x_2 x - y_2 y) + i(y_2 x + x_2 y)$$

or, on equating real and imaginary parts,

$$\begin{aligned} x_2x - y_2y &= x_1 \\ y_2x + x_2y &= y_1 \end{aligned}$$

or

$$\begin{bmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \tag{4}$$

Since $z_2 = x_2 + iy_2 \neq 0$, it follows that x_2 and y_2 are not both zero, so

$$\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix} = x_2^2 + y_2^2 \neq 0$$

Thus, by Cramer's rule (Theorem 2.1.4), system 4 has the unique solution

$$x = \frac{\begin{vmatrix} x_1 & -y_2 \\ y_1 & x_2 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{|z_2|^2}$$

$$y = \frac{\begin{vmatrix} x_2 & x_1 \\ y_2 & y_1 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} = \frac{y_1x_2 - x_1y_2}{|z_2|^2}$$

Therefore,

$$\begin{aligned} z = x + iy &= \frac{1}{|z_2|^2}[(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)] \\ &= \frac{1}{|z_2|^2}(x_1 + iy_1)(x_2 - iy_2) = \frac{1}{|z_2|^2} z_1\bar{z}_2 \end{aligned}$$

■

Thus, for $z_2 \neq 0$, we define

$$\boxed{\frac{z_1}{z_2} = \frac{1}{|z_2|^2} z_1\bar{z}_2} \tag{5}$$

Remark To remember this formula, multiply the numerator and denominator of z_1 / z_2 by $\bar{z}_2$:

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{1}{|z_2|^2} z_1\bar{z}_2$$

EXAMPLE 3 Quotient in the Form $a+bi$

Express

$$\frac{3+4i}{1-2i}$$

in the form $a+bi$.

Solution

From 5 with $z_1=3+4i$ and $z_2=1-2i$,

$$\begin{aligned}\frac{3+4i}{1-2i} &= \frac{1}{|1-2i|^2}(3+4i)(\overline{1-2i}) = \frac{1}{5}(3+4i)(1+2i)\\ &= \frac{1}{5}(-5+10i) = -1+2i\end{aligned}$$

Alternative Solution

As in the remark above, multiply numerator and denominator by the conjugate of the denominator:

$$\frac{3+4i}{1-2i} = \frac{3+4i}{1-2i}\cdot\frac{1+2i}{1+2i} = \frac{-5+10i}{5} = -1+2i$$

Systems of linear equations with complex coefficients arise in various applications. Without going into detail, we note that all the results about linear systems studied in Chapters 1 and 2 carry over without change to systems with complex coefficients. Note, however, that a few results studied in other chapters *will* change for complex matrices.

EXAMPLE 4 A Linear System with Complex Coefficients

Use Cramer's rule to solve

$$\begin{aligned} ix+2y &= 1-2i\\ 4x-iy &= -1+3i\end{aligned}$$

Solution

$$x = \frac{\begin{vmatrix} 1-2i & 2 \\ -1+3i & -i \end{vmatrix}}{\begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix}} = \frac{(-i)(1-2i)-2(-1+3i)}{i(-i)-2(4)} = \frac{-7i}{-7} = i$$

$$y = \frac{\begin{vmatrix} i & 1-2i \\ 4 & -1+3i \end{vmatrix}}{\begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix}} = \frac{(i)(-1+3i)-4(1-2i)}{i(-i)-2(4)} = \frac{-7+7i}{-7} = 1-i$$

Thus the solution is $x = i$, $y = 1 - i$.

We conclude this section by listing some properties of the complex conjugate that will be useful in later sections.

THEOREM 10.2.3

Properties of the Conjugate

For any complex numbers z, z_1, and z_2:

(a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(b) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(c) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(d) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$

(e) $\bar{\bar{z}} = z$

We prove (*a*) and leave the rest as exercises.

Proof (a) Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$; then

$$\begin{aligned}\overline{z_1 + z_2} &= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\ &= (a_1 + a_2) - (b_1 + b_2)i \\ &= (a_1 - b_1 i) + (a_2 - b_2 i) \\ &= \bar{z}_1 + \bar{z}_2\end{aligned}$$

■

Remark It is possible to extend part (*a*) of Theorem 10.2.3 to *n* terms and part (*c*) to *n* factors. More precisely,

$$\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$$
$$\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n$$

Exercise Set 10.2

Click here for Just Ask!

1. In each part, find $\bar{z}$.

(a) $z = 2 + 7i$

(b) $z = -3 - 5i$

(c) $z = 5i$

(d) $z = -i$

(e) $z = -9$

(f) $z = 0$

2. In each part, find $|z|$.

(a) $z = i$

(b) $z = -7i$

(c) $z = -3 - 4i$

(d) $z = 1 + i$

(e) $z = -8$

(f) $z = 0$

3. Verify that $z\bar{z} = |z|^2$ for

(a) $z = 2 - 4i$

(b) $z = -3 + 5i$

(c) $z = \sqrt{2} - \sqrt{2}i$

4. Given that $z_1 = 1 - 5i$ and $z_2 = 3 + 4i$, find

(a) z_1 / z_2

(b) $\bar{z}_1 / z_2$

(c) $z_1 / \bar{z}_2$

(d) $\overline{(z_1 / z_2)}$

(e) $z_1 / |z_2|$

(f) $|z_1 / z_2|$

5. In each part, find $1 / z$.

(a) $z = i$

(b) $z = 1 - 5i$

(c) $z = \frac{-i}{7}$

6. Given that $z_1 = 1 + i$ and $z_2 = 1 - 2i$, find

(a) $z_1 - \left(\frac{z_1}{z_2}\right)$

(b) $\frac{z_1 - 1}{z_2}$

(c) $z_1^2 - \left(\frac{iz_1}{z_2}\right)$

(d) $\dfrac{z_1}{iz_2}$

In Exercises 7–14 perform the calculations and express the result in the form $a+bi$.

7. $\dfrac{i}{1+i}$

8. $\dfrac{2}{(1-i)(3+i)}$

9. $\dfrac{1}{(3+4i)^2}$

10. $\dfrac{2+i}{i(-3+4i)}$

11. $\dfrac{\sqrt{3}+i}{(1-i)(\sqrt{3}-i)}$

12. $\dfrac{1}{i(3-2i)(1+i)}$

13. $\dfrac{i}{(1-i)(1-2i)(1+2i)}$

14. $\dfrac{1-2i}{3+4i}-\dfrac{2+i}{5i}$

15. In each part, solve for z.

(a) $iz=2-i$

(b) $(4-3i)\overline{z}=i$

16. Use Theorem 10.2.3 to prove the following identities:

(a) $\overline{\overline{z}+5i}=z-5i$

(b) $\overline{iz}=-i\overline{z}$

(c) $\dfrac{\overline{i+\overline{z}}}{i-z}=-1$

17. In each part, sketch the set of points in the complex plane that satisfies the equation.

(a) $|z|=2$

(b) $|z-(1+i)|=1$

(c) $|z-i|=|z+i|$

(d) $\text{Im}\,(\overline{z}+i)=3$

18. In each part, sketch the set of points in the complex plane that satisfies the given condition(s).

(a) $|z+i|\leq 1$

(b) $1<|z|<2$

(c) $|2z-4i|<1$

(d) $|z|\leq|z+i|$

19. Given that $z = x + iy$, find

(a) $\operatorname{Re}(\overline{iz})$

(b) $\operatorname{Im}(\overline{iz})$

(c) $\operatorname{Re}(i\bar{z})$

(d) $\operatorname{Im}(i\bar{z})$

20.

(a) Show that if n is a positive integer, then the only possible values for $(1/i)^n$ are 1, −1, i, and $-i$.

(b) Find $(1/i)^{2509}$.

Hint See Exercise 23(b) of Section 10.1.

21. Prove:

(a) $\frac{1}{2}(z + \bar{z}) = \operatorname{Re}(z)$

(b) $\frac{1}{2i}(z - \bar{z}) = \operatorname{Im}(z)$

22. Prove: $z = \bar{z}$ if and only if z is a real number.

23. Given that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$, find

(a) $\operatorname{Re}\left(\frac{z_1}{z_2}\right)$

(b) $\operatorname{Im}\left(\frac{z_1}{z_2}\right)$

24. Prove: If $(\bar{z})^2 = z^2$, then z is either real or pure imaginary.

25. Prove that $|z| = |\bar{z}|$.

26. Prove:

(a) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(c) $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2$

(d) $\bar{\bar{z}} = z$

27.

(a) Prove that $\overline{z^2} = (\bar{z})^2$.

(b) (b) Prove that if n is a positive integer, then $\overline{z^n} = (\bar{z})^n$.

(c) Is the result in part (b) true if n is a negative integer? Explain.

In Exercises 28–31 solve the system of linear equations by Cramer's rule.

28. $$\begin{aligned} ix_1 - ix_2 &= -2 \\ 2x_1 + x_2 &= i \end{aligned}$$

29. $$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2i \end{aligned}$$

30. $$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 2 + 2i \\ x_1 - x_2 + x_3 &= -1 \end{aligned}$$

31. $$\begin{aligned} ix_1 + 3x_2 + (1+i)x_3 &= -i \\ x_1 + ix_2 + 3x_3 &= -2i \\ x_1 + x_2 + x_3 &= 0 \end{aligned}$$

In Exercises 32 and 33 solve the system of linear equations by Gauss–Jordan elimination.

32. $$\begin{bmatrix} -1 & -1-i \\ -1+i & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

33. $$\begin{bmatrix} 2 & -1-i \\ -1+i & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

34. Solve the following system of linear equations by Gauss–Jordan elimination.

$$\begin{aligned} x_1 + ix_2 - ix_3 &= 0 \\ -x_1 + (1-i)x_2 + 2ix_3 &= 0 \\ 2x_1 + (-1+2i)x_2 - 3ix_3 &= 0 \end{aligned}$$

35. In each part, use the formula in Theorem 1.4.5 to compute the inverse of the matrix, and check your result by showing that $AA^{-1} = A^{-1}A = I$.

(a) $A = \begin{bmatrix} i & -2 \\ 1 & i \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & i \\ 1 & 0 \end{bmatrix}$

36. Let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial for which the coefficients a_0, a_1, a_2, …, a_n are real. Prove that if z is a solution of the equation $p(z) = 0$, then so is $\bar{z}$.

37. Prove: For any complex number z, $|\operatorname{Re}(z)| \leq |z|$ and $|\operatorname{Im}(z)| \leq |z|$.

38. Prove that

$$\frac{|\operatorname{Re}(z)| + |\operatorname{Im}(z)|}{\sqrt{2}} \leq |z|$$

Hint Let $z = x + iy$ and use the fact that $(|x| - |y|)^2 \geq 0$.

39. In each part, use the method of Example 4 in Section 1.5 to find A^{-1}, and check your result by showing that $AA^{-1} = A^{-1}A = I$.

(a) $$A = \begin{bmatrix} 1 & 1+i & 0 \\ 0 & 1 & i \\ -i & 1-2i & 2 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} i & 0 & -i \\ 0 & 1 & -1-4i \\ 2-i & i & 3 \end{bmatrix}$$

40. Show that $|z - 1| = |\bar{z} - 1|$. Discuss the geometric interpretation of the result.

41.

(a) If $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, find $|z_1 - z_2|$ and interpret the result geometrically.

(b) Use part (a) to show that the complex numbers 12, $6 + 2i$, and $8 + 8i$ are vertices of a right triangle.

42. Use Theorem 10.2.3 to show that if the coefficients *a*, *b*, and *c* in a quadratic polynomial are real, then the solutions of the equation $az^2 + bz + c = 0$ are complex conjugates. What can you conclude if *a*, *b*, and *c* are complex?

Print this page

10.3 POLAR FORM OF A COMPLEX NUMBER

In this section we shall discuss a way to represent complex numbers using trigonometric properties. Our work will lead to an important formula for powers of complex numbers and to a method for finding nth roots of complex numbers.

Polar Form

If $z = x + iy$ is a nonzero complex number, $r = |z|$, and θ measures the angle from the positive real axis to the vector z, then, as suggested by Figure 10.3.1,

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{1}$$

so that $z = x + iy$ can be written as $z = r\cos\theta + ir\sin\theta$ or

$$z = r(\cos\theta + i\sin\theta) \tag{2}$$

This is called a ***polar form of z***.

Argument of a Complex Number

The angle θ is called an ***argument of z*** and is denoted by

$$\theta = \arg z$$

The argument of z is not uniquely determined because we can add or subtract any multiple of 2π from θ to produce another value of the argument. However, there is only one value of the argument in radians that satisfies

$$-\pi < \theta \leq \pi$$

This is called the ***principal argument of z*** and is denoted by

$$\theta = \operatorname{Arg} z$$

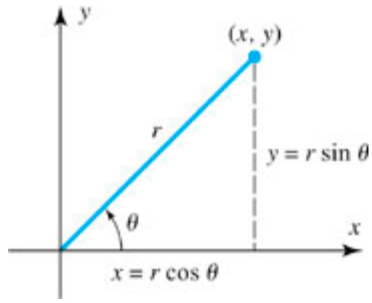


Figure 10.3.1

EXAMPLE 1 Polar Forms

Express the following complex numbers in polar form using their principal arguments:

(a) $z = 1 + \sqrt{3}i$

(b) $z = -1 - i$

Solution (a)

The value of r is

$$r = |z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

and since $x = 1$ and $y = \sqrt{3}$, it follows from 1 that

$$1 = 2\cos\theta \qquad \text{and} \qquad \sqrt{3} = 2\sin\theta$$

so $\cos\theta = 1/2$ and $\sin\theta = \sqrt{3}/2$. The only value of θ that satisfies these relations and meets the requirement $-\pi < \theta \le \pi$ is $\theta = \pi/3 (= 60°)$ (see Figure 10.3.2*a*). Thus a polar form of z is

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Solution (b)

The value of r is

$$r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

and since $x = -1$, $y = -1$, it follows from 1 that

$$-1 = \sqrt{2}\cos\theta \qquad \text{and} \qquad -1 = \sqrt{2}\sin\theta$$

so $\cos\theta = -1/\sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. The only value of θ that satisfies these relations and meets the requirement $-\pi < \theta \le \pi$ is $\theta = -3\pi/4 (= -135°)$ (Figure 10.3.2*b*). Thus, a polar form of z is

$$z = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$$

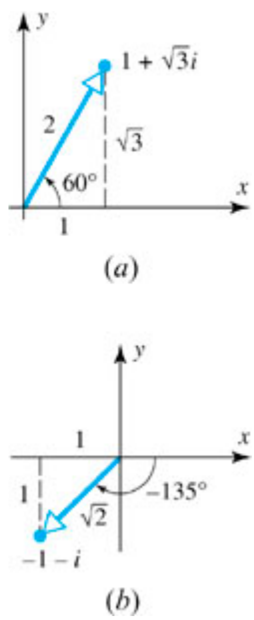


Figure 10.3.2

Multiplication and Division Interpreted Geometrically

We now show how polar forms can be used to give geometric interpretations of multiplication and division of complex numbers. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad \text{and} \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Multiplying, we obtain

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

Recalling the trigonometric identities

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

we obtain

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \qquad (3)$$

which is a polar form of the complex number with modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$. Thus we have shown that

$$|z_1 z_2| = |z_1||z_2| \tag{4}$$

and

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

(Why?) In words, *the product of two complex numbers is obtained by multiplying their moduli and adding their arguments* (Figure 10.3.3).

We leave it as an exercise to show that if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \tag{5}$$

from which it follows that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{if } z_2 \neq 0$$

and

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

In words, *the quotient of two complex numbers is obtained by dividing their moduli and subtracting their arguments* (*in the appropriate order*).

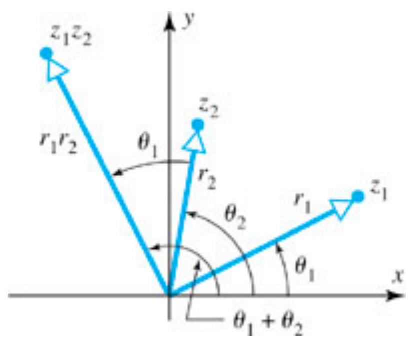


Figure 10.3.3 The product of two complex numbers.

EXAMPLE 2 A Quotient Using Polar Forms

Let

$$z_1 = 1 + \sqrt{3}i \qquad \text{and} \qquad z_2 = \sqrt{3} + i$$

Polar forms of these complex numbers are

$$z_1 = 2\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right) \qquad \text{and} \qquad z_2 = 2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

(verify) so that from 3,

$$z_1z_2 = 4\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$
$$= 4\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right] = 4[0+i] = 4i$$

and from 5,

$$\frac{z_1}{z_2} = 1\cdot\left[\cos\left(\frac{\pi}{3}-\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\right]$$
$$= \cos\frac{\pi}{6}+i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}+\frac{1}{2}i$$

As a check, we calculate z_1z_2 and z_1 / z_2 directly without using polar forms for z_1 and z_2:

$$z_1z_2 = (1+\sqrt{3}i)(\sqrt{3}+i) = (\sqrt{3}-\sqrt{3})+(3+1)i = 4i$$
$$\frac{z_1}{z_2} = \frac{1+\sqrt{3}i}{\sqrt{3}+i}\cdot\frac{\sqrt{3}-i}{\sqrt{3}-i} = \frac{(\sqrt{3}+\sqrt{3})+(-i+3i)}{4} = \frac{\sqrt{3}}{2}+\frac{1}{2}i$$

which agrees with our previous results.

The complex number i has a modulus of 1 and an argument of $\pi / 2 (= 90°)$, so the product iz has the same modulus as z, but its argument is 90° greater than that of z. In short, *multiplying z by i rotates z counterclockwise by* 90° (Figure 10.3.4).

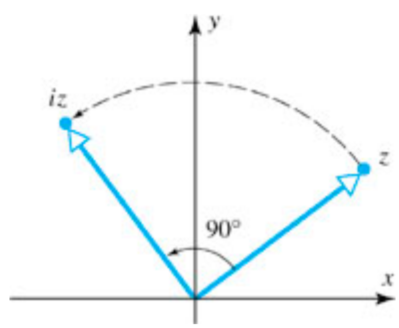


Figure 10.3.4

Multiplying by i rotates z counterclockwise by 90°.

DeMoivre's Formula

If n is a positive integer and $z = r(\cos\theta + i\sin\theta)$, then from Formula 3,

$$z^n = \underbrace{z\cdot z\cdot\cdots z}_{n-\text{factors}} = r^n[\cos\underbrace{(\theta+\theta+\cdots+\theta)}_{n-\text{terms}} + i\sin\underbrace{(\theta+\theta+\cdots+\theta)}_{n-\text{terms}}]$$

or

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{6}$$

Moreover, 6 also holds for negative integers if $z \neq 0$ (see Exercise 23).

In the special case where $r = 1$, we have $z = \cos\theta + i\sin\theta$, so 6 becomes

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{7}$$

which is called ***DeMoivre's formula***. Although we derived 7 assuming *n* to be a positive integer, it will be shown in the exercises that this formula is valid for all integers *n*.

Finding *n*th Roots

We now show how DeMoivre's formula can be used to obtain roots of complex numbers. If *n* is a positive integer and *z* is any complex number, then we define an ***nth root of z*** to be any complex number *w* that satisfies the equation

$$w^n = z \tag{8}$$

We denote an *n*th root of *z* by $z^{1/n}$. If $z \neq 0$, then we can derive formulas for the *n*th roots of *z* as follows. Let

$$w = \rho(\cos\alpha + i\sin\alpha) \qquad \text{and} \qquad z = r(\cos\theta + i\sin\theta)$$

If we assume that *w* satisfies 8, then it follows from 6 that

$$\rho^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta) \tag{9}$$

Comparing the moduli of the two sides, we see that $\rho^n = r$ or

$$\rho = \sqrt[n]{r}$$

where $\sqrt[n]{r}$ denotes the real positive *n*th root of *r*. Moreover, in order to have the equalities $\cos n\alpha = \cos\theta$ and $\sin n\alpha = \sin\theta$ in 9, the angles $n\alpha$ and θ must either be equal or differ by a multiple of 2π. That is,

$$n\alpha = \theta + 2k\pi \qquad \text{or} \qquad \alpha = \frac{\theta}{n} + \frac{2k\pi}{n}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Thus the values of $w = \rho(\cos\alpha + i\sin\alpha)$ that satisfy 8 are given by

$$w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \qquad k = 0, \pm 1, \pm 2, \ldots$$

Although there are infinitely many values of *k*, it can be shown (see Exercise 16) that $k = 0$, 1, 2, …, $n - 1$ produce distinct values of *w* satisfying 8 but all other choices of *k* yield duplicates of these. Therefore, there are exactly *n* different *n*th roots of $z = r(\cos\theta + i\sin\theta)$, and these are given by

$$z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \qquad k = 0, 1, 2, \ldots, n - 1 \tag{10}$$

Abraham DeMoivre *(1667–1754)* was a French mathematician who made important contributions to probability, statistics, and trigonometry. He developed the concept of statistically independent events, wrote a major and influential treatise on probability, and helped transform trigonometry from a branch of geometry into a branch of analysis through his use of complex numbers. In spite of his important work, he barely managed to eke out a living as a tutor and a consultant on gambling and insurance.

EXAMPLE 3 Cube Roots of a Complex Number

Find all cube roots of −8.

Solution

Since −8 lies on the negative real axis, we can use $\theta = \pi$ as an argument. Moreover, $r = |z| = |-8| = 8$, so a polar form of −8 is

$$-8 = 8(\cos \pi + i \sin \pi)$$

From 10 with $n = 3$, it follows that

$$(-8)^{1/3} = \sqrt[3]{8}\left[\cos\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right], \qquad k = 0,\ 1,\ 2$$

Thus the cube roots of −8 are

$$2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = 1 + \sqrt{3}i$$

$$2(\cos \pi + i \sin \pi) = 2(-1) = -2$$

$$2 \cos\left(\frac{5\pi}{3} + i \sin \frac{5\pi}{3}\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 1 - \sqrt{3}i$$

As shown in Figure 10.3.5, the three cube roots of −8 obtained in Example 3 are equally spaced $\pi / 3$ radians $(= 120°)$ apart around the circle of radius 2 centered at the origin. This is not accidental. In general, it follows from Formula 10 that the *n*th roots of *z* lie on the circle of radius $\sqrt[n]{r}(= \sqrt[n]{|z|})$ and are equally spaced $2\pi / n$ radians apart. (Can you see why?) Thus, once one *n*th root of *z* is found, the remaining $n - 1$ roots can be generated by rotating this root successively through increments of $2\pi / n$ radians.

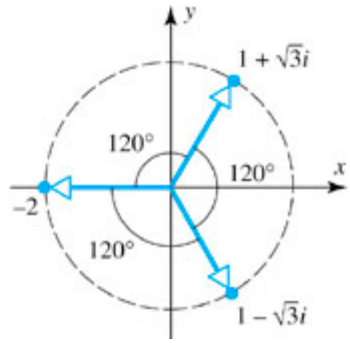


Figure 10.3.5

The cube roots of −8.

EXAMPLE 4 Fourth Roots of a Complex Number

Find all fourth roots of 1.

Solution

We could apply Formula 10. Instead, we observe that $w = 1$ is one fourth root of 1, so the remaining three roots can be generated by rotating this root through increments of $2\pi / 4 = \pi / 2$ radians $(= 90°)$. From Figure 10.3.6, we see that the fourth roots of 1 are

$$1, \quad i, \quad -1, \quad -i$$

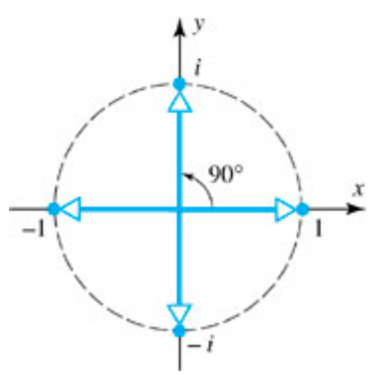


Figure 10.3.6

The fourth roots of 1.

Complex Exponents

We conclude this section with some comments on notation.

In more detailed studies of complex numbers, complex exponents are defined, and it is shown that

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{11}$$

where e is an irrational real number given approximately by $e \approx 2.71828\ldots$. (For readers who have studied calculus, a proof of this result is given in Exercise 18.)

It follows from 11 that the polar form

$$z = r(\cos\theta + i\sin\theta)$$

can be written more briefly as

$$z = re^{i\theta} \tag{12}$$

EXAMPLE 5 Expressing a Complex Number in Form 12

In Example 1 it was shown that

$$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

From 12 this can also be written as

$$1 + \sqrt{3}i = 2e^{i\pi/3}$$

◆

It can be proved that complex exponents follow the same laws as real exponents, so if

$$z_1 = r_1e^{i\theta_1} \quad \text{and} \quad z_2 = r_2e^{i\theta_2}$$

are nonzero complex numbers, then

$$z_1z_2 = r_1r_2e^{i\theta_1 + i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i\theta_1 - i\theta_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$$

But these are just Formulas 3 and 5 in a different notation.

We conclude this section with a useful formula for $\bar{z}$ in polar notation. If

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

then

$$\bar{z} = r(\cos\theta - i\sin\theta) \tag{13}$$

Recalling the trigonometric identities

$$\sin(-\theta) = -\sin\theta \qquad \text{and} \qquad \cos(-\theta) = \cos\theta$$

we can rewrite 13 as

$$\bar{z} = r[\cos(-\theta) + i\sin(-\theta)] = re^{i(-\theta)}$$

or, equivalently,

$$\bar{z} = re^{-i\theta} \tag{14}$$

In the special case where $r = 1$, the polar form of z is $z = e^{i\theta}$, and 14 yields the formula

$$\overline{e^{i\theta}} = e^{-i\theta} \tag{15}$$

Exercise Set 10.3

Click here for Just Ask!

1. In each part, find the principal argument of z.

(a) $z = 1$

(b) $z = i$

(c) $z = -i$

(d) $z = 1 + i$

(e) $z = -1 + \sqrt{3}i$

(f) $z = 1 - i$

2. In each part, find the value of $\theta = \arg(1 - \sqrt{3}i)$ that satisfies the given condition.

(a) $0 < \theta \leq 2\pi$

(b) $-\pi < \theta \leq \pi$

(c) $-\frac{\pi}{6} \leq \theta < \frac{11\pi}{6}$

3. In each part, express the complex number in polar form using its principal argument.

(a) $2i$

(b) -4

(c) $5 + 5i$

(d) $-6 + 6\sqrt{3}i$

(e) $-3 - 3i$

(f) $2\sqrt{3} - 2i$

4. Given that $z_1 = 2(\cos \pi / 4 + i \sin \pi / 4)$ and $z_2 = 3(\cos \pi / 6 + i \sin \pi / 6)$, find a polar form of

(a) $z_1 z_2$

5. Express $z_1 = i$, $z_2 = 1 - \sqrt{3}i$, and $z_3 = \sqrt{3} + i$ in polar form, and use your results to find $z_1 z_2 / z_3$. Check your results by performing the calculations without using polar forms.

6. Use Formula 6 to find

(a) $(1 + i)^{12}$

(b) $\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)^{-6}$

(c) $(\sqrt{3} + i)^7$

(d) $(1 - i\sqrt{3})^{-10}$

7. In each part, find all the roots and sketch them as vectors in the complex plane.

(a) $(-i)^{1/2}$

(b) $(1 + \sqrt{3}i)^{1/2}$

(c) $(-27)^{1/3}$

(d) $(i)^{1/3}$

(e) $(-1)^{1/4}$

(f) $(-8 + 8\sqrt{3}i)^{1/4}$

8. Use the method of Example 4 to find all cube roots of 1.

9. Use the method of Example 4 to find all sixth roots of 1.

10. Find all square roots of $1+i$ and express your results in polar form.

11. Find all solutions of the equation $z^4-16=0$.

12. Find all solutions of the equation $z^4+8=0$ and use your results to factor z^4+8 into two quadratic factors with real coefficients.

13. It was shown in the text that multiplying *z* by *i* rotates *z* counterclockwise by 90°. What is the geometric effect of dividing *z* by *i*?

14. In each part, use 6 to calculate the given power.

(a) $(1+i)^8$

(b) $(-2\sqrt{3}+2i)^{-9}$

15. In each part, find $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$.

(a) $z=3e^{i\pi}$

(b) $z=3e^{-i\pi}$

(c) $\bar{z}=\sqrt{2}e^{\pi i/2}$

(d) $\bar{z} = -3e^{-2\pi i}$

16.

(a) Show that the values of $z^{1/n}$ in Formula 10 are all different.

(b) Show that integer values of *k* other than $k = 0$, 1, 2, …, $n - 1$ produce values of $z^{1/n}$ that are duplicates of those in Formula 10.

17. Show that Formula 7 is valid if $n = 0$ or *n* is a negative integer.

18. **(For Readers Who Have Studied Calculus)** To prove Formula 11, recall that the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

(a) By substituting $x = i\theta$ in this series and simplifying, show that

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

(b) Use the result in part (a) to obtain Formula 11.

19. Derive Formula 5.

20. When $n = 2$ and $n = 3$, Equation 7 gives

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$
$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

Use these two equations to obtain trigonometric identities for $\cos 2\theta$, $\sin 2\theta$, $\cos 3\theta$, and $\sin 3\theta$.

21. Use Formula 11 to show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

22. Show that if $(a + bi)^3 = 8$, then $a^2 + b^2 = 4$.

23. Show that Formula 6 is valid for negative integer exponents if $z \neq 0$.

Answers to Selected Questions
from *Elementary Linear Algebra: Applications Version, 9th ed.*

Chapter 10.1

#5. **a)** $2 + 3i$, **b)** $-1-2i$, **c)** $-2+9i$, **#11**. $76 - 88i$, **#17.** 0,

#19. **a)** $\begin{bmatrix} 1+6i & -3+7i \\ 3+8i & 3+12i \end{bmatrix}$, **b)** $\begin{bmatrix} 3-2i & 6+5i \\ 3-5i & 13+3i \end{bmatrix}$, **c)** $\begin{bmatrix} 3+3i & 2+5i \\ 9-5i & 13-2i \end{bmatrix}$, **d)** $\begin{bmatrix} 9+i & 12+2i \\ 18-2i & 13+i \end{bmatrix}$

#22. $-1 \pm i$, $\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Chapter 10.2

#9. $-\frac{7}{625} - \frac{24}{625} i$ **#11.** $\frac{1-\sqrt{3}}{4} + \frac{1+\sqrt{3}}{4} i$ **#15.** **a)** $-1-2i$, **b)** $-3/25-4i/25$,

#19. **a)** $-y$, **b)** $-x$, **c)** y, **d)** x, **#33.** $(1 + i)t, 2t$, **#35.** **a)** $\begin{bmatrix} i & 2 \\ -1 & i \end{bmatrix}$, **b)** $\begin{bmatrix} 0 & 1 \\ -i & 2i \end{bmatrix}$

Chapter 10.3

#3. **a)** $2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$, **b)** $4\left(\cos(\pi) + i\sin(\pi)\right)$,

c) $5\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$, **d)** $12\left(\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right)$

e) $3\sqrt{2}\left(\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right)$ **f)** $4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

#5. $z_1 = e^{i\pi}$, $z_2 = 2e^{-i\pi/3}$, $z_3 = 2e^{i\pi/6}$, $\frac{z_1 z_2}{z_3} = 1$

#7. b) $\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i$, $-\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i$ **#11**. ± 2, $\pm 2i$

c) -3, $\frac{3}{2} + \frac{3\sqrt{3}}{2} i$, $\frac{3}{2} - \frac{3\sqrt{3}}{2} i$

#15. a) $\mathrm{Re}(z) = -3$, $\mathrm{Im}(z) = 0$
b) $\mathrm{Re}(z) = -3$, $\mathrm{Im}(z) = 0$
c) $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = -\sqrt{2}$
d) $\mathrm{Re}(z) = -3$, $\mathrm{Im}(z) = 0$