# Continuous model theory and the classification problem

Bradd Hart
McMaster University

Joint work with Ilijas Farah

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# Outline

- Basics of C*-algebras
- Reminder about continuous model theory
- Basic model theory of operator algebras
- The Elliott classification problem
- More advanced C*-algebra basics
- A model theory conjecture

# C*-algebra basics

## Definition

A C*-algebra is a *-subalgebra $A$ of the bounded linear operators $B(H)$ on a complex Hilbert space $H$ which is closed in the operator norm topology. Alternatively, a C*-algebra is a Banach *-algebra $A$ which satisfies the C*-identity $\|a^*a\| = \|a\|^2$ for all $a \in A$.

The first sentence defines a concrete representation of a C*-algebra and the second gives an abstract definition.

## Theorem (Gel'fand, Naimark, Sigal)

*Every abstract C*-algebra has a concrete representation.*

Examples:

- $M_n(\mathbb{C})$; in general, $B(H)$; $C_0(X)$ for any locally compact space $X$ - these form all the commutative C*-algebras.
- C*-algebras are closed under inductive limits where the relevant morphisms are *-homomorphisms.
- C*-algebras are closed under tensor products but ...

# Spectral Theorem

## Definition

If $A$ is a unital C*-algebra and $a \in A$ then $sp(a)$, the spectrum of $a$, is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda I$ is not invertible.

## Theorem (Spectral Theorem)

*Suppose $A$ is a unital C*-algebra and $a \in A$ is self-adjoint ($a^* = a$) then $C^*(a)$, the C*-subalgebra of $A$ generated by $a$ and $I$ is isomorphic to $C(sp(a))$ via the map which sends $a$ to the identity and $I$ to $1$.*

Example: If $A$ is a C*-algebra and $p \in A$, we call $p$ a projection if $p^2 = p$ $(= p^*)$.
Claim: For every $\epsilon > 0$ there is a $\delta > 0$ such that if $a$ is self-adjoint and $\|a^2 - a\| < \delta$ then there is a projection $p$ such that $\|p - a\| < \epsilon$.

# Continuous model theory of C*-algebras

- A C*-algebra can be thought of as a metric structure by introducing a sort for each ball of operator norm $N \in \mathbb{N}$.
- One has function symbols for the sorted operations of $+, \cdot$ and $*$ as well as the unary operations of multiplication by $\lambda$ for every $\lambda \in \mathbb{C}$. It is sometimes useful to consider an expanded language in which one has a function symbol for every *-polynomial (again properly sorted).
- The only relation symbol is the operator norm $\| \cdot \|$.
- The basic formulas of continuous logic which are relevant here are $\|p(\bar{x})\|$ where $p(\bar{x})$ is a *-polynomial.
- Formulas are closed under composition with continuous real-valued functions; moreover, if $\varphi$ is a formula then so is $\sup_{x \in B_N} \varphi$ or $\inf_{x \in B_N} \varphi$. The interpretation of these formulas in a C*-algebra is standard.

# The theory of C*-algebras

- Notice the if $A$ is a C*-algebra, $\bar{a} \in A$ and $\varphi$ is a formula then $\varphi^A(\bar{a})$ is a number. In particular, if $\varphi$ is a sentence then $\varphi^A \in \mathbb{R}$.
- $Th(A)$, the theory of an algebra, is the function which to every sentence $\varphi$ assigns $\varphi^A$. A theory is determined by its zero set.
- We say that a class of structures $K$ is elementary if there is a set of sentences $T$ such that $A \in K$ iff $\varphi^A = 0$ for all $\varphi \in T$.

## Theorem

*The class of C*-algebras is an elementary class. In fact, in the appropriate language it is a universal class.*

# Ultraproducts

- If $A_i$ for $i \in I$ are C*- algebras and $U$ is an ultrafilter on $I$, one forms the norm ultraproduct as follows:
- Let
$$\ell^\infty(\prod_{i \in I} A_i) = \{\bar{a} \in \prod_{i \in I} A_i : \text{for some } M, \|a_i\| \leq M \text{ for all } i \in I\}$$
and
$$c_U = \{\bar{a} \in \ell^\infty(\prod_{i \in I} A_i) : \lim_{i \to U} \|a_i\| = 0\}$$
- The ultraproduct is then $\prod_{i \in I} A_i / U := \ell^\infty(\prod_{i \in I} A_i)/c_U$.

# Definable zero sets

## Definition

Suppose that $M$ is a metric structure and $\varphi(\bar{x})$ is a formula. We say that $\varphi$ has a definable zero set if the distance function to the zero set of $\varphi$, $\{\bar{a} \in M : \varphi^M(\bar{a}) = 0\}$, is given by a definable predicate in $M$ i.e. a uniform limit of formulas.

## Theorem

*For a metric structure $M$ and a formula $\varphi$, the following are equivalent:*

- *$\varphi$ has a definable zero set.*
- *The zero set of $\varphi$ can be quantified over.*

# Stable relations

## Definition

In the language of C*-algebras, a formula $\varphi(\bar{x})$, or its zero set, is called a stable relation if for every C*-algebra $A$ and for every $\epsilon > 0$ there is a $\delta > 0$ such that if $\bar{a} \in A$ and $|\varphi(\bar{a})| < \delta$ then there is $\bar{b} \in A$ such that $\varphi(\bar{b}) = 0$ and $\|\bar{a} - \bar{b}\| < \epsilon$.

## Lemma

*Among C*-algebras, the notions of stable relation and definable zero set are the same.*

Examples of stable relations:

- the set of projections.
- the sets of self-adjoint elements, unitary elements ($u^*u = uu^* = 1$), positive elements ($a^*a$); in general, the range of any term.
- the sets of generators for subalgebras isomorphic to $M_n(C)$, for any $n \in \mathbb{N}$ or, in general, any finite-dimensional algebra.

# The classification programme for nuclear C*-algebras

## The Elliott conjecture

The isomorphism type of a simple, separable, infinite-dimensional, unital nuclear C*-algebra is determined by its K-theory.

- For a C*-algebra $A$, there is an invariant called the Elliott invariant which for the record is defined as:

$$Ell(A) = ((K_0(A), K_0^+(A), [1_A]), K_1(A), Tr(A), \rho_A)$$

- There are other invariants which come up like KK-theory and the Cuntz semi-group but I won't focus on them.

# Nuclear algebras

## Definition

A C*-algebra $A$ is called nuclear if for all C*-algebras $B$, $A\bar{\otimes}B$ is uniquely defined.

Examples:

- All abelian C*-algebras are nuclear.
- $M_n(\mathbb{C})$ is nuclear but $B(H)$ for an infinite-dimensional Hilbert space $H$ is not nuclear.
- The class of nuclear algebras is closed under tensor products hence $M_n(C(X))$ is nuclear for any compact space $X$.
- The class of nuclear algebras is closed under inductive limits; UHF (uniformly hyperfinite) algebras are limits of matrix algebras; AF (approximately finite dimensional) algebras are limits of finite-dimensional algebras.
- The class of nuclear algebras is not closed under ultraproducts or even ultrapowers.

# Nuclear algebras, cont'd

## Definition

- A element of a C*-algebra $A$ is said to be positive if it is of the form $a^*a$ for some $a \in A$.
- A linear map $f : A \to B$ is positive if whenever $a \in A$ is positive then so is $f(a)$.
- A linear map $f : A \to B$ is completely positive if the induced map from $M_n(A)$ to $M_n(B)$ is positive for all $n$.
- A map $f$ is contractive if $\|f\| \leq 1$.

## Theorem (Stinespring)

*For any completely positive map $f : A \to B(H)$ there is a Hilbert space $K$, *-homomorphism $\pi : A \to B(K)$ and $V \in B(K, H)$ such that $f(a) = V\pi(a)V^*$.*

# Nuclear algebras: good news and bad news

## Definition

A C*-algebra $A$ has the contractive positive approximation property (CPAP) if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an $n$ and cpc maps $\sigma : A \to M_n(\mathbb{C})$ and $\tau : M_n(\mathbb{C}) \to A$ such that $\|\bar{a} - \tau(\sigma(\bar{a}))\| < \epsilon$.

## Theorem (Choi-Effros, Kirchberg)

*A C*-algebra A is nuclear iff it satisfies the CPAP.*

## Theorem

*There are countably many partial types such that a C*-algebra is nuclear iff it omits all of these types.*

# The definition of $K_0$

## Definition

For any C*-algebra $A$, consider the equivalence relation $\sim$ on projections in $A$ given by $p \sim q$ iff there is some $v \in A$, $vpv^* = q$ and $v^*qv = p$.

Consider the (non-unital) *-homomorphism $\Phi_n : M_n(A) \to M_{n+1}(A)$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and let $M_\infty = \lim_n M_n(A)$. We should really complete this ...

Let $V(A) = Proj(M_\infty(A))/\sim$.

$V(A)$ has an additive structure defined as follows: if $p, q \in V(A)$ then $p \oplus q$ is

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

# The definition of $K_0$, cont'd

## Definition

$K_0(A)$ is the Grothendieck group generated from $(V(A), \oplus)$ and $K_0^+(A)$ is the image of $V(A)$ in $K_0(A)$; if $A$ is unital then the constant $[1_A]$ corresponds to the identity in $A$.

Examples:

- $K_0(M_n(\mathbb{C}))$ is $(\mathbb{Z}, \mathbb{N}, n)$.
- If $H$ is infinite-dimensional then $K_0(B(H))$ is 0.
- Consider $A = \lim_n M_{2^n}(\mathbb{C})$ where the given morphisms are $M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C})$ such that

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

  Then $K_0(A)$ is the dyadic rationals with the unit associated to 1.

# Examples of $K_0$, cont'd

- In general, if $A = \lim_k M_{n(k)}$ where $n(k)|n(k+1)$ for all $k$ and the morphisms are given by diagonal maps

$$a \mapsto \begin{pmatrix} a & & & 0 \\ & a & & \\ & & \ddots & \\ 0 & & & a \end{pmatrix}$$

then $K_0(A) = \{m/n : m \in \mathbb{Z} \text{ and } n|n(k) \text{ for some } k\}$.

# The main actors in K-theory for nuclear C*-algebras

- We have already introduced $(K_0(A), K_0^+(A), [1_A])$.
- $K_1(A) = K_0(C_0((0,1), A))$.
- $Tr(A)$ is the set of traces on $A$ i.e. all positive linear functionals $\tau$ on $A$ such that $\tau(1) = 1$, $\tau(x^*) = \overline{\tau(x)}$ and $\tau(xy) = \tau(yx)$.
- $\rho_A$ is the natural pairing of $Tr(A)$ and $K_0(A)$.
- The form of the Elliott conjecture which states that the Elliott invariant classifies all simple, separable, infinite-dimensional, unital nuclear algebras is false - there are counter-examples of different types with the first ones due to Toms and separately Rørdam.
- A search is on for a new invariant which might classify nuclear algebras.

# Prototypical example of classification

## Theorem (Elliott)

*The class of AF algebras can be classified by $K_0$.*

Let's do a special case of this result due to Glimm.

## Definition

For a UHF algebra $A = \lim_k M_{n(k)}$, let the $GI(A)$, the generalized integer of $A$ be the function which assigns to every prime $p$ the supremum of all $n$ such that $p^n$ divides $n(k)$ for some $k$; this can be infinite.

## Theorem (Glimm)

*If $A$ and $B$ are separable, unital UHF algebras then $A \cong B$ iff $GI(A) = GI(B)$.*

# A proof of Glimm's theorem

Sketch of proof: One checks that UHF algebras have a unique trace and the values of this trace on a UHF algebra $A$ are of the form $\{k/n : k \in \mathbb{N}, n | Gl(A)\}$.

Now if $Gl(A) = Gl(B)$ then we can arrange in a back and forth fashion that $A = \lim_k M_{n(k)}$ and $B = \lim_k M_{m(k)}$ such that for all $k$, $n(k)|m(k)|n(k+1)$. It is possible then to create a sequence of maps $\varphi_k : M_{n(k)} \to M_{m(k)}$ and $\psi_k : M_{m(k)} \to M_{n(k+1)}$ which additionally have the necessary commutation to make $A$ and $B$ isomorphic.

$$
\begin{array}{ccccccccc}
M_{n(1)}(\mathbb{C}) & \longrightarrow & M_{n(2)}(\mathbb{C}) & \longrightarrow & M_{n(3)}(\mathbb{C}) & \longrightarrow & \cdots & & A \\
\phi_1 \downarrow & \psi_1 \nearrow & \phi_2 \downarrow & \psi_2 \nearrow & \phi_3 \downarrow & & & & \\
M_{m(1)}(\mathbb{C}) & \longrightarrow & M_{m(2)}(\mathbb{C}) & \longrightarrow & M_{m(3)}(\mathbb{C}) & \longrightarrow & \cdots & & B
\end{array}
$$

# Model theoretic version of the Elliott conjecture

*Simple, separable, infinite-dimensional, unital nuclear algebras are classified by their Elliott invariant and their first order continuous theory.*

# $K_0(A)$ vs. $Th(A)$, round 1

- In the case of a separable, unital UHF algebra $A$, $K_0(A)$ is a rank 1, torsion-free abelian group where we have specified a constant. This is determined by $GI(A)$ by Glimm's theorem.
- Equivalently, the theory knows the generalized integer for a separable, unital UHF algebra $A = \lim_k M_{m(k)}$. In fact, $M_n$ embeds into $A$ iff $n$ divides $n(k)$ for some $k$.
- Round 1 - a draw.

# $K_0(A)$ vs. $Th(A)$, round 2

- A classical result of Dixmier which generalizes Glimm's theorem shows that non-unital separable UHF algebras are classified by $K_0$.
- In this case, $K_0$ is an arbitrary rank 1, torsion-free abelian group.
- The isomorphism relation for such groups is known not to be smooth in the sense of Borel equivalence relations.
- The theory of a C*-algebra is a smooth invariant and so Dixmier's result shows that $K_0$ and not the theory captures isomorphism at least for non-unital separable UHF algebras.
- Advantage $K_0$ (and descriptive set theory).

# K-theory vs. $Th(A)$, round 3

- The most general counter-examples to the form of the Elliott conjecture which says that $Ell(A)$ is a sufficient invariant are due to Toms, Annals of Math, 2008.
- He gave continuum many simple separable nuclear C*-algebras with identical Elliott invariant that were not isomorphic.
- He used something called the Cuntz semigroup to show they were not isomorphic and in particular computed a number called the radius of comparison - it was this value that differentiated the algebras.
- In joint work with Leonel Robert, we showed that the radius of comparison is known to the theory of an algebra - it is preserved under ultraproducts and elementary submodels.
- Advantage $Th(A)$.

# Traces matter

- Nuclear algebras do not form an elementary class but it is interesting to consider the theory of nuclear algebras.
- Question: is every C*-algebra elementarily equivalent to a nuclear algebra?
- No. Let $A = \prod_{n \in \mathbb{N}} M_n(C)/U$ where $U$ is a non-principal ultrafilter on $\mathbb{N}$.
- We need some facts about $A$: $A$ has a trace and it is definable say by a formula $\varphi$.
- Now suppose that $A \equiv B$ where $B$ is some simple, separable, nuclear algebra.

# Traces matter, cont'd

- But then $\varphi$ would define a trace on $B$ which would mean that the associated von Neumann algebra is the hyperfinite $\mathrm{II}_1$ factor $\mathcal{R}$.
- In earlier work with Farah and Sherman we showed that a property identified by von Neumann called property $\Gamma$ for tracial von Neumann algebras was elementary.
- It is known that $\mathcal{R}$ satisfies property $\Gamma$ and that $A$ modulo its trace does not so $A \not\equiv B$.
- Question: what is the theory of the class of nuclear algebras? Is it the theory of C*-algebras?