Revisiting classification theory from the 1970s

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Outline

- Basics of C*-algebras
- Reminder about continuous model theory
- Basic model theory of operator algebras
- The Elliott classification problem
- The classification of AF algebras
- Some model theoretic variants

C*-algebra basics

Definition

A C*-algebra is a *-subalgebra A of the bounded linear operators $B(H)$ on a complex Hilbert space H which is closed in the operator norm topology. Alternatively, a C*-algebra is a Banach *-algebra A which satisfies the C*-identity $\|a^* a\| = \|a\|^2$ for all $a \in A$.

The first sentence defines a concrete representation of a C*-algebra and the second gives an abstract definition.

Theorem (Gel'fand, Naimark, Sigal)

Every abstract C-algebra has a concrete representation.*

Examples:

- $M_n(\mathbb{C})$; in general, $B(H)$; $C_0(X)$ for any locally compact space X - these form all the commutative C*-algebras.
- C*-algebras are closed under inductive limits where the relevant morphisms are *-homomorphisms.
- C*-algebras are closed under tensor products but ...

Continuous model theory of C*-algebras

- A C*-algebra can be thought of as a metric structure by introducing a sort for each ball of operator norm $N \in \mathbb{N}$.
- One has function symbols for the sorted operations of $+, \cdot$ and $*$ as well as the unary operations of multiplication by λ for every $\lambda \in \mathbb{C}$. It is sometimes useful to consider an expanded language in which one has a function symbol for every *-polynomial (again properly sorted).
- The only relation symbol is the operator norm $\| \cdot \|$.
- The basic formulas of continuous logic which are relevant here are $\|p(\bar{x})\|$ where $p(\bar{x})$ is a *-polynomial.
- Formulas are closed under composition with continuous real-valued functions; moreover, if φ is a formula then so is $\sup_{x \in B_N} \varphi$ or $\inf_{x \in B_N} \varphi$. The interpretation of these formulas in a C*-algebra is standard.

The theory of C*-algebras

- Notice the if A is a C*-algebra, $\bar{a} \in A$ and φ is a formula then $\varphi^A(\bar{a})$ is a number. In particular, if φ is a sentence then $\varphi^A \in \mathbb{R}$.
- $Th(A)$, the theory of an algebra, is the function which to every sentence φ assigns φ^A. A theory is determined by its zero set on non-negative sentences.
- We say that a class of structures K is elementary if there is a set of non-negative sentences T such that $A \in K$ iff $\varphi^A = 0$ for all $\varphi \in T$.

Theorem

The class of C-algebras is an elementary class. In fact, in the appropriate language it is a universal class.*

The classification programme for nuclear C*-algebras

The Elliott programme

Determine which simple, separable, infinite-dimensional nuclear C*-algebra are determined by their K-theory.

- For a C*-algebra A, there is an invariant called the Elliott invariant which for the record is defined as:

$$Ell(A) = ((K_0(A), K_0^+(A), [1_A]), K_1(A), Tr(A), \rho_A)$$

- There are other invariants which come up like KK-theory and the Cuntz semi-group but I won't focus on them.

Nuclear algebras

Definition

A C*-algebra A is called nuclear if for all C*-algebras B, $A\bar{\otimes}B$ is uniquely defined.

Examples:

- All abelian C*-algebras are nuclear.
- $M_n(\mathbb{C})$ is nuclear but $B(H)$ for an infinite-dimensional Hilbert space H is not nuclear.
- The class of nuclear algebras is closed under tensor products hence $M_n(C(X))$ is nuclear for any compact space X.
- The class of nuclear algebras is closed under inductive limits; UHF (uniformly hyperfinite) algebras are limits of matrix algebras; AF (approximately finite dimensional) algebras are limits of finite-dimensional algebras.
- The class of nuclear algebras is not closed under ultraproducts or even ultrapowers.

Nuclear algebras, cont'd

Definition

- A element of a C*-algebra A is said to be positive if it is of the form a^*a for some $a \in A$.
- A linear map $f : A \to B$ is positive if whenever $a \in A$ is positive then so is $f(a)$.
- A linear map $f : A \to B$ is completely positive if the induced map from $M_n(A)$ to $M_n(B)$ is positive for all n.
- A map f is contractive if $\|f\| \leq 1$.

Theorem (Stinespring)

*For any completely positive map $f : A \to B(H)$ there is a Hilbert space K, *-homomorphism $\pi : A \to B(K)$ and $V \in B(K, H)$ such that $f(a) = V\pi(a)V^*$.*

Nuclear algebras: good news and bad news

Definition

A C*-algebra A has the contractive positive approximation property (CPAP) if for every $\bar{a} \in A$ and $\epsilon > 0$ there is an n and cpc maps $\sigma : A \to M_n(\mathbb{C})$ and $\tau : M_n(\mathbb{C}) \to A$ such that $\|\bar{a} - \tau(\sigma(\bar{a}))\| < \epsilon$.

Theorem (Choi-Effros, Kirchberg)

A C-algebra A is nuclear iff it satisfies the CPAP.*

Theorem

There are countably many partial types such that a C-algebra is nuclear iff it omits all of these types.*

AF algebras

- An AF algebra is an inductive limit of finite-dimensional C*-algebras.
- All finite-dimensional C*-algebras are isomorphic to finite direct sums of matrix algebras.
- Suppose that $nk \leq m$. Define the map $\varphi_k : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ such that $\varphi_k(A) =$

$$\begin{pmatrix} A & \dots & & & 0 \\ 0 & A & \dots & & 0 \\ 0 & 0 & \ddots & & 0 \\ 0 & \dots & & A & \dots \\ 0 & & \dots & & 0 \end{pmatrix}$$

  where A appears k times along the diagonal.
- If f is any *-homomorphism from M_n to M_m then f is unitarily equivalent to φ_k for some k.
- Any *-homomorphism between AF algebras is understood via its Bratteli diagram (see picture) which determines the homomorphism up to unitary conjugation.

The definition of K_0

Definition

For any C*-algebra A, consider the equivalence relation $\sim$ on projections in A given by $p \sim q$ iff there is some $v \in A$, $vpv^* = q$ and $v^*qv = p$.

Consider the (non-unital) *-homomorphism $\Phi_n : M_n(A) \to M_{n+1}(A)$ defined by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

and let $M_\infty = \lim_n M_n(A)$. We should really complete this ...

Let $V(A) = Proj(M_\infty(A))/\sim$.

$V(A)$ has an additive structure defined as follows: if $p, q \in V(A)$ then $p \oplus q$ is

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

The definition of K_0, cont'd

Definition

$K_0(A)$ is the Grothendieck group generated from $(V(A), \oplus)$ and $K_0^+(A)$ is the image of $V(A)$ in $K_0(A)$; if A is unital then the constant $[1_A]$ corresponds to the identity in A.

Examples:

- $K_0(M_n(\mathbb{C}))$ is $(\mathbb{Z}, \mathbb{N}, n)$.
- If H is infinite-dimensional then $K_0(B(H))$ is 0.
- Consider $A = \lim_n M_{2^n}(\mathbb{C})$ where the given morphisms are $M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C})$ such that

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

  Then $K_0(A)$ is the dyadic rationals with the unit associated to 1.

Properties of K_0

- K_0 is a functor from the category of C*-algebras with *-homomorphisms into the category of ordered abelian groups.
- K_0 commutes with direct sums and inductive limits.
- So for any AF algebra A, the underlying abelian group is the limit of groups of the form Z^n for various n's.

Prototypical example of classification

Theorem (Elliott)

The class of AF algebras can be classified by K_0 i.e. if A and B are separable AF algebras and $K_0(A) \cong K_0(B)$ then $A \cong B$.

A sketch of the proof:

Model theoretic versions of the Elliott conjecture

- What is the relationship between $K_0(A)$ and $Th(A)$ when A is an AF algebra?
- A classical result of Dixmier shows that non-unital separable UHF algebras are classified by K_0.
- In this case, K_0 is an arbitrary rank 1, torsion-free abelian group.
- The isomorphism relation for such groups is known not to be smooth in the sense of Borel equivalence relations.
- The theory of a C*-algebra is a smooth invariant and so Dixmier's result shows that K_0 and not the theory captures isomorphism at least for non-unital separable UHF algebras.
- Crazy conjecture: for AF algebras A and B, if $K_0(A) \equiv K_0(B)$ then $A \equiv B$.
- We don't know of a single concrete example where A and B are non-isomorphic, elementarily equivalent separable AF algebras.

Model theoretic versions of the Elliott conjecture

- Crazy conjecture 2: Simple, separable, infinite-dimensional, unital nuclear algebras are classified by their Elliott invariant and their first order continuous theory.
- The evidence for this is almost non-existent.
- The most general counter-examples to the form of the Elliott conjecture which says that $Ell(A)$ is a sufficient invariant are due to Toms, Annals of Math, 2008.
- He gave continuum many simple separable nuclear C*-algebras with identical Elliott invariant that were not isomorphic.
- He used something called the Cuntz semigroup to show they were not isomorphic and in particular computed a number called the radius of comparison - it was this value that differentiated the algebras.
- We showed that the radius of comparison is known to the theory of an algebra - it is preserved under ultraproducts and elementary submodels.