Model theory and the fundamental lemma

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Overview

- The Langland’s program

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- The fundamental lemma

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- Motivic integration

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- Transfer principles

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- Results of Cluckers and Loeser

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- Langland conjectured, among other things, that for a number field F, there was a relationship, a specific correspondence, between n-dimensional Galois representations of F and automorphic representations of $GL_n(\mathbb{A}_F)$.

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- Ngô Bao Châu, Le Lemme Fondamental Pour Les Algèbres De Lie, ArXiv, May, 2008.

1. The naked lemma

Suppose G to be an unramified reductive group defined over a $\mathfrak{p}$-adic field $\mathfrak{k}$, H an unramified endoscopic group of G. The two have the same rank.

Given a strongly regular semi-simple element t of H, let T_t be its centalizer in H, a maximal torus. There exists a special embedding of T_t in G. The element t gives rise to an element t_G of G, which we assume to be strongly regular in G.

Define orbital integrals of a locally constant function of compact support on any reductive group—for any strongly regular t let

$$\Lambda(f,t) = \Big(\prod_{\Sigma_G} |\alpha(t) - 1|^{1/2}\Big) \left|\frac{\operatorname{vol} T(\mathfrak{o})}{\operatorname{vol} G(\mathfrak{o})}\right| \int_{C_G(t)\backslash G} f(g^{-1}tg)\,\frac{dg}{dt}\,.$$

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The ratio of volumes is to eliminate the effect of choice of measures on G and T.

Orbital integrals have something to with fixed-points, since $C_G(t)$ is the set of points on G fixed under conjugation by t. We shall see that this connection extends to something very deep. The product is over roots α with respect to T. Such factors are familiar in fixed point formulas.

The data determining H have something to do a set of elements s in the torus $\widehat{T}$ in the L-group ${}^LG$. They all give rise to the same map κ from the stable conjugacy class of t_G, constant on the ordinary conjugacy classes. The κ-orbital integrals are linear combinations of the ordnary orbital integrals:

$$\Lambda^{\kappa}_{G,H}(f,t) = \sum_{t' \sim t} \kappa(t')\Lambda_G(f,t')\,.$$

If $\kappa \equiv 1$ this is the stable sum $\Lambda^{\mathrm{st}}_G(f,t)$.

Stable conjugacy means conjugacy in the algebrac closure.

Any element f of the Hecke algebra $\mathcal{H}\big(G//G(\mathfrak{o})\big)$ gives rise to an f^H in $\mathcal{H}(H//H(\mathfrak{o}))$.

Fundamental Lemma:

$$\Lambda^{\kappa}_{G/H}(f, t_G) = \Lambda^{\mathrm{st}}_{H}(f^H, t)$$

for all strongly G-regular semi-simple elements t of H.

Very roughly, this says that analysis on G invariant under conjugation can be reduced to stable analysis on G and all its endoscopic groups.

Implicit in this are assertions about character sums in G matching stable character sums in H.

In this way, each cocycle a_τ gives a complex constant $\kappa(a_\tau) \in \mathbb{C}^\times$.

Example 5.5. The element $s \in \mathbb{C}^\times$ giving the endoscopic group $H = U_E(1)$ of $SL(2)$ is $s = -1$, which may be identified with the character $n \mapsto (-1)^n$ of $\mathbb{Z}$. This gives the nontrivial character κ of

$$H^1(\mathrm{Gal}(\bar{F}/F), U_E(1)) \cong \mathbb{Z}/2\mathbb{Z}.$$

6. Statement of the Fundamental Lemma

6.1. **Context.** Let G be an unramified connected reductive group over F. Let H be an unramified endoscopic group of G. Let $\gamma \in H(F)$ be a strongly regular semisimple element. Let $T_H = C_H(\gamma)$, and let T_G be a Cartan subgroup of G that is isomorphic to it. More details will be given below about how to choose T_G. The choice of T_G matters! Let $\gamma \in T_H(F)$ map to $\gamma_0 \in T_G(F)$ under this isomorphism.

By construction, γ_0 is semisimple. However, as G may have more roots than H, it is possible for γ_0 to be singular, even when γ is strongly regular. If $\gamma \in H(F)$ is a strongly regular semisimple element with the property that γ_0 is also strongly regular, then we will call γ a *strongly G-regular* element of $H(F)$.

If γ' is stably conjugate to γ_0 with cocycle a_τ, then $s \in \mathrm{Hom}(X_*, \mathbb{C}^\times)$ gives $\kappa(a_\tau) \in \mathbb{C}^\times$.

Let K_G and K_H be hyperspecial maximal compact subgroups of G and H. Let $\chi_{G,K}$ and $\chi_{H,K}$ be the characteristic functions of these hyperspecial subgroups. Set

(6.0.1)

$$\Lambda_{G,H}(\gamma) = \left(\prod_{\alpha\in\Phi_G} |\alpha(\gamma_0)-1|^{1/2}\right)\left[\frac{\mathrm{vol}(K_T,dt)}{\mathrm{vol}(K,dg)}\right]\sum_{\gamma'\sim\gamma_0}\kappa(a_\tau)\int_{C_G(\gamma',F)\backslash G(F)}\chi_{G,K}(g^{-1}\gamma' g)\frac{dg}{dt'}.$$

The set of roots Φ_G are taken to be those relative to T_G. The sum runs over all stable conjugates γ' of γ_0, up to conjugacy. This is a finite sum. The group K_T is defined to be the maximal compact subgroup of T_G. Equation 6.0.1 is a finite linear combination of orbital integrals (that is, integrals over conjugacy classes in the group with respect to an invariant measure). The Haar measures dt' on $C_G(\gamma', F)$ and dt on $T_G(F)$ are chosen so that stable conjugacy between the two groups is measure preserving. This particular linear combination of integrals is called a κ-orbital integral because of the term $\kappa(a_\tau)$ that gives the coefficients of the linear combination. Note that the integration takes place in the group G, and yet the parameter γ is an element of $H(F)$.

The volume terms $\mathrm{vol}(K, dg)$ and $\mathrm{vol}(K_T, dt)$ serve no purpose other than to make the entire expression independent of the choice of Haar measures dg and dt, which are only defined up to a scalar multiple.

We can form an analogous linear combination of orbital integrals on the group H. Set

(6.0.2)

$$\Lambda_H^{st}(\gamma) = \left(\prod_{\alpha\in\Phi_H} |\alpha(\gamma)-1|^{1/2}\right)\left[\frac{\mathrm{vol}(K_T,dt)}{\mathrm{vol}(K_H,dh)}\right]\sum_{\gamma'\sim\gamma}\int_{C_H(\gamma',F)\backslash H(F)} \chi_{H,K}(h^{-1}\gamma' h)\frac{dh}{dt'}.$$

This linear combination of integrals is like $\Lambda_{G,H}(\gamma)$, except that H replaces G, K_H replaces K_G, Φ_H (taken relative to T_H) replaces Φ_G, and so forth. Also, the factor $\kappa(a_\tau)$ has been dropped. The linear combination of Equation 6.0.2 is called a stable orbital integral, because it extends over all stable conjugates of the element γ without the factor κ. The superscript *st* in the notation is for 'stable.'

Conjecture 6.1. *(The fundamental lemma) For every $\gamma \in H(F)$ that is strongly G-regular semisimple,*

$$\Lambda_{G,H}(\gamma) = \Lambda_H^{st}(\gamma).$$

Remark 6.2. There have been serious efforts over the past twenty years to prove the fundamental lemma. These efforts have not yet led to a proof. Thus, the fundamental lemma is not a lemma; it is a conjecture with a misleading name. Its name leads one to speculate that the authors of the conjecture may have severely underestimated the difficulty of the conjecture.

Remark 6.3. Special cases of the fundamental lemma have been proved. The case $G = SL(n)$ was proved by Waldspurger [28]. Building on the work of [5], Laumon has proved that the fundamental lemma for $G = U(n)$ follows from a purity conjecture [21]. The fundamental lemma has not been proved for any other general families of groups. The fundamental lemma has been proved for some groups G of small rank, such as $SU(3)$ and $Sp(4)$. See [2], [7], [10].

6.2. **The significance of the fundamental lemma.** The Langlands program predicts correspondences $\pi \leftrightarrow \pi'$ between the representation theory of different reductive groups. There is a local program for the representation theory of reductive groups over locally compact fields, and a global program for *automorphic* representations of reductive groups over the adele rings of global fields.

The Arthur-Selberg trace formula has emerged as a powerful tool in the Langlands program. In crude terms, one side of the trace formula contains terms related to the characters of automorphic representations. The other side contains terms such as orbital integrals. *Thanks to the trace formula, identities between orbital integrals on different groups imply identities between the representations of the two groups.*

It is possible to work backwards: from an analysis of the terms in the trace formula and a precise conjecture in representation theory, it is possible to make precise conjectures about identities of orbital integrals. The most basic identity that appears in this way is the fundamental lemma, articulated above.

The proofs of many major theorems in automorphic representation theory depend in one way or another on the proof of a fundamental lemma. For example, the proof of Fermat's Last Theorem depends on Base Change for $GL(2)$, which in turn depends on the fundamental lemma for cyclic base change [17]. The proof of the local Langlands conjecture for $GL(n)$ depends on automorphic induction, which in turn depends on the fundamental lemma for $SL(n)$ [11], [12], [28]. Properties of the zeta function of Picard modular varieties depend on the fundamental lemma for $U(3)$ [26], [2]. Normally, the dependence of a major theorem on a particular lemma would not be noteworthy. It is only because the fundamental lemma has not been proved in general, and because the lack of proof has become a serious impediment to progress in the field, that the conjecture has become the subject of increased scrutiny.

7. Reductions

To give a trivial example of the fundamental lemma, if γ and γ_0 and their stable conjugates are not in any compact subgroup, then

$$\chi_{G,K}(g^{-1}\gamma' g) = 0 \text{ and } \chi_{H,K}(h^{-1}\gamma' h) = 0$$

so that both $\Lambda_{G,H}(\gamma)$ and $\Lambda^{st}_H(\gamma)$ are zero. Thus, the fundamental lemma holds for trivial reasons for such γ.

7.1. **Topological Jordan decomposition.**

A somewhat less trivial reduction of the problem is provided by the topological Jordan decomposition. Suppose that γ lies in a compact subgroup. It can be written uniquely as a product

$$\gamma = \gamma_s\gamma_u = \gamma_u\gamma_s,$$

where γ_s has finite order, of order prime to the residue field characteristic p, and γ_u is topologically unipotent. That is,

$$\lim_{n\to\infty} \gamma_u^{p^n} = 1.$$

The limit is with respect to the p-adic topology. A special case of the topological Jordan decomposition $\gamma \in O_F^\times \subset \mathbb{G}_m(F)$ is treated in [13, p20]. In that case, γ_s is defined by the formula

$$\gamma_s = \lim_{n\to\infty} \gamma^{q^n}.$$

Let γ, γ_0, and γ' be chosen as in Section 6.1. Each of these elements has a topological Jordan decomposition. Let $G_s = C_G(\gamma_{0s})$ and $H_s = C_H(\gamma_s)$. It turns out that G_s is an unramified reductive group with unramified endoscopic group H_s. Descent for orbital integrals gives the formulas [20] [8]

$$\begin{aligned} \Lambda_{G,H}(\gamma) &= \Lambda_{G_s,H_s}(\gamma_u) \\ \Lambda^{st}_H(\gamma) &= \Lambda^{st}_{H_s}(\gamma_u). \end{aligned}$$

LE LEMME FONDAMENTAL POUR LES ALGÈBRES DE LIE

par

Ngô Bao Châu

Introduction

Dans cet article, nous proposons une démonstration pour des conjectures de Langlands, Shelstad et Waldspurger plus connues sous le nom de lemme fondamental pour les algèbres de Lie et lemme fondamental non standard. On se reporte à 1.11.1 et à 1.12.7 pour plus de précisions dans les énoncés suivants.

Théorème 1. — *Soient k un corps fini à q éléments, $\mathcal{O}$ un anneau de valuation discrète complet de corps résiduel k et F son corps des fractions. Soit G un schéma en groupes réductifs au-dessus de $\mathcal{O}$ dont l'ordre du groupe de Weyl n'est pas divisible par la caractéristique de k. Soient (κ, ρ_κ) une donnée endoscopique de G au-dessus de $\mathcal{O}$ et H le schéma en groupes endoscopiques associé.*

On a l'égalité entre la κ-intégrale orbitale et l'intégrale orbitale stable

$$\Delta_G(a)\mathbf{O}^\kappa_a(1_{\mathfrak{g}}, \mathrm{d}t) = \Delta_H(a_H)\mathbf{SO}_{a_H}(1_{\mathfrak{h}}, \mathrm{d}t)$$

associées aux classes de conjugaison stable semi-simples régulières a et a_h de $\mathfrak{g}(F)$ et $\mathfrak{h}(F)$ qui se correspondent, aux fonctions caractéristiques $1_{\mathfrak{g}}$ et $1_{\mathfrak{h}}$ des compacts $\mathfrak{g}(\mathcal{O})$ et $\mathfrak{h}(\mathcal{O})$ dans $\mathfrak{g}(F)$ et $\mathfrak{h}(F)$ et où on a noté

$$\Delta_G(a) = q^{-\mathrm{val}(\mathfrak{D}_G(a))/2} \text{ et } \Delta_H(a_H) = q^{-\mathrm{val}(\mathfrak{D}_H(a_H))/2}$$

$\mathfrak{D}_G$ et $\mathfrak{D}_H$ étant les fonctions discriminant de G et de H.

Théorème 2. — *Soient G_1, G_2 deux schémas en groupes réductifs sur $\mathcal{O}$ ayant des données radicielles isogènes dont l'ordre du groupe de Weyl n'est pas divisible par la caractéristique de k. Alors, on a l'égalité suivante entre les intégrales orbitales stables*

$$\mathbf{SO}_{a_1}(1_{\mathfrak{g}_1}, \mathrm{d}t) = \mathbf{SO}_{a_2}(1_{\mathfrak{g}_2}, \mathrm{d}t)$$

associées aux classes de conjugaison stable semi-simples régulières a_1 et a_2 de $\mathfrak{g}_1(F)$ et $\mathfrak{g}_2(F)$ qui se correspondent et aux fonctions caractéristiques $1_{\mathfrak{g}_1}$ et $1_{\mathfrak{g}_2}$ des compacts $\mathfrak{g}_1(\mathcal{O})$ et $\mathfrak{g}_2(\mathcal{O})$ dans $\mathfrak{g}_1(F)$ et $\mathfrak{g}_2(F)$.

Nous démontrons ces théorèmes dans le cas d'égale caractéristique. D'après Waldspurger, le cas d'inégales caractéristiques s'en déduit *cf.* [**78**].

Les applications principales du lemme fondamental se trouvent dans la réalisation de certains cas particuliers du principe de fonctorialité de Langlands via la comparaison de formules des traces et dans la construction de représentations galoisiennes attachées aux formes automorphes par le biais du calcul de cohomologie des variétés de Shimura. On se réfère aux travaux d'Arthur [**2**] pour les applications à la comparaison de formules des traces et à l'article de Kottwitz [**42**] ainsi qu'au livre en préparation édité par Harris pour les applications aux variétés de Shimura.

Cas connus et réductions. — Le lemme fondamental a été établi dans un grand nombre de cas particuliers. Son analogue archimédien a été entièrement résolu par Shelstad dans [**67**]. Ce cas a incité Langlands et Shelstad à formuler leur conjecture pour un corps non-archimédien. Le cas du groupe SL(2) a été traité par Labesse et Langlands dans [**46**]. Le cas du groupe unitaire à trois variables a été résolu par Rogawski dans [**62**]. Les cas assimilés aux Sp(4) et GL(4) tordu ont été résolus par Hales, Schröder et Weissauer par des calculs explicites *cf.* [**30**], [**64**] et [**81**]. Récemment, Whitehouse a poursuivi ces calculs pour démontrer le lemme fondamental pondéré tordu dans ce cas *cf.* [**80**].

Le lemme fondamental pour le changement de base stable a été établi par Clozel [**12**] et Labesse [**45**] à partir du cas de l'unité de l'algèbre de Hecke démontré par Kottwitz [**40**]. Auparavant, le cas GL(2) a été établi par Langlands [**47**] et le cas GL(3) par Kottwitz [**36**].

Un autre cas important est le cas SL(n) résolu par Waldspurger dans [**75**]. Le cas SL(3) avec un tore elliptique a été établi auparavant par

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- Some basics: The Denef-Pas language, $\mathcal{L}_{\mathcal{DP}}$ for the ring of integers $\mathcal{O}$ for some number field k which is sometimes written $\mathcal{L}_{\mathcal{O}}$.

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- R. Cluckers, A course on motivic integration, ModNet tutorial in LaRoche, Apr. 2008.

For each $N > 0$ and for each group homomorphism

$$\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$$

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consider the sum

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with f a definable function in the ring language.

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How do these finite sums $S_{\psi,N,\varphi,f}$ vary with $N > 0$,

and how do they vary in definable families?

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Still more generally,

$$S_{\psi,\chi,N,\varphi,f,g} := \sum_{x \in \varphi(\mathbb{Z}/N\mathbb{Z})} \psi(f(x))\chi(g(x)),$$

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where "basic" often means that N is only allowed to be a prime number, One of the quests of Motivic Integration can be described as the search for a as wide as possible class of formulas φ and definable functions f, g, so that for as many as possible numbers $N > 0$ and choices of the characters φ, χ, the dependence of $S_{\psi,\chi,N,\varphi,f,g}$ on N (or of similar even more general objects) (and the dependence in

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"basic objects" of a similar kind $S_{\psi,\chi,N,\varphi,f,g}$,

where "basic" often means that N is only allowed to be a prime number,

or even better: in terms of an abstract, "basic" geometric object.

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This quest of Motivic Integration tries to be as uniform as possible in number fields as well.

- From finite sums to integration on local fields

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Henselian local fields are:

$$\mathbb{F}_q((t))$$

and finite field extensions of

$$\mathbb{Q}_p,$$

the p-adic completion of $\mathbb{Q}$ for the norm $|p^\ell a/b|_p = p^{-\ell}$, $\ell \in \mathbb{Z}$.

Back to (definable) integration

Let $\psi : \mathbb{Q}_p \to \mathbb{C}^\times$ be an additive character

Back to (definable) integration

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ψ can vary, as well as χ_a, a, and $\mathbb{Q}_p$.

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- How does $I_{(\cdot)}$ depend on ψ, χ_a, and a?
- More importantly, how does it depend on $\mathbb{Q}_p$?
- How does it vary in definable families?

The char p analogue

Likewise,
Let $\psi : \mathbb{F}_p((t)) \to \mathbb{C}^\times$ be an additive character which is trivial on $p\mathbb{F}_p[[t]]$ and nontrivial on $\mathbb{F}_p[[t]]$.

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New question:
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or even better: in terms of abstract, "basic" geometric objects.

Transfer principles

The original transfer principle: Ax-Kochen-Ershov

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More slides from Cluckers' LaRoche tutorial and results from Cluckers, Hale and Loeser, A transfer principle for the fundamental lemma, ArXiv, Dec. 2007.

In terms of these basic objects or of (°°°), the theory understands the integrals

$$I_{\varphi,K} := \int_{\{x \in K^n | \varphi(x)\}} \psi(f)|dx|,$$

when K varies over Henselian local fields of big enough residual characteristic.
And similarly in definable families using any kinds and any number of parameters and quantifiers!

However, by the very nature of ψ, which has a different range (image) when the local field varies (not only depending on the residue field), this integral does not only depend on the residue field,
and thus, a naive version of the transfer principle makes no sense.

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holds, only depends on the residue field of K.

The Cluckers - Loeser generalisation of the transfer principle

This transfer principle (with parameter dependence and with ψ) is very promising for applications in the Langlands program and representation theory.
The application to several variants of the Fundamental Lemma has been worked out recently (see C. - Hales - Loeser) and to aspects of representation theory by Gordon and Cunningham.

$h[1,0,0]$. We denote by $f : Z \to Y$ the morphism induced by projection. Then $[\mathbf{1}_Z]$ is S-integrable if and only if $\mathbb{L}^{(\mathrm{ordjac}\, f)\circ f^{-1}}$ is, and then $f_!([\mathbf{1}_Z]) = \mathbb{L}^{(\mathrm{ordjac}\, f)\circ f^{-1}}$.

Once Theorem 2.5.1 is proved, one may proceed as follows to extend the constructions from C_+ to C. One defines $\mathrm{I}_S C(Z)$ as the subgroup of $C(Z)$ generated by the image of $\mathrm{I}_S C_+(Z)$. One shows that if $f : Z \to Y$ is a morphism in Def_S, the morphism $f_! : \mathrm{I}_S C_+(Z) \to \mathrm{I}_S C_+(Y)$ has a natural extension

$$f_! : \mathrm{I}_S C(Z) \to \mathrm{I}_S C(Y). \tag{2.5.3}$$

The proof of Theorem 2.5.1 is quite long and involved. In a nutshell, the basic idea is the following. Integration along residue field variables is controlled by (A5) and integration along $\mathbb{Z}$-variables by (A6). Integration along valued field variables is constructed one variable after the other. To integrate with respect to one valued field variable, one may, using (a variant of) the cell decomposition Theorem 2.2.1 (at the cost of introducing additional new residue field and $\mathbb{Z}$-variables), reduce to the case of cells which is covered by (A7) and (A8). An important step is to show that this is independent of the choice of a cell decomposition. When one integrates with respect to more than one valued field variable (one after the other) it is crucial to show that it is independent of the order of the variables, for which we use a notion of bicells.

2.6. **Motivic measure.** The relation of Theorem 2.5.1 with motivic integration is the following. When S is equal to $h[0,0,0]$, the final object of Def_k, one writes $\mathrm{I}C_+(Z)$ for $\mathrm{I}_S C_+(Z)$ and we shall say integrable for S-integrable, and similarly for C. Note that $\mathrm{I}C_+(h[0,0,0]) = C_+(h[0,0,0]) = SK_0(\mathrm{RDef}_k) \otimes_{\mathbb{N}[\mathbb{L}-1]} A_+$ and that $\mathrm{I}C(h[0,0,0]) = K_0(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} A$. For φ in $\mathrm{I}C_+(Z)$, or in $\mathrm{I}C(Z)$, one defines the motivic integral $\mu(\varphi)$ by $\mu(\varphi) = f_!(\varphi)$ with f the morphism $Z \to h[0,0,0]$.

Let X be in Def_k of dimension d. Let φ be a function in $\mathscr{C}_+(X)$, or in $\mathscr{C}(X)$. We shall say φ is integrable if its class $[\varphi]_d$ in $C_+^d(X)$, resp. in $C^d(X)$, is integrable, and we shall set

$$\mu(\varphi) = \int_X \varphi \, d\mu = \mu([\varphi]_d).$$

Using the following Change of Variables Theorem 2.6.1, one may develop the integration on global (non affine) objects endowed with a differential form of top degree (similarly as in the p-adic case), cf. [7].

Theorem 2.6.1 (Cluckers-Loeser [7])**.** *Let $f : Y \to X$ be an isomorphism in* Def_k. *For any integrable function φ in $\mathscr{C}_+(X)$ or $\mathscr{C}(X)$,*

$$\int_X \varphi d\mu = \int_Y \mathbb{L}^{-\mathrm{ord}\,\mathrm{jac}(f)} f^*(\varphi) d\mu.$$

Also, the construction we outlined of the motivic measure carries over almost literally to a relative setting: one can develop a relative theory of motivic integration: integrals depending on parameters of functions in $\mathscr{C}_+$ or $\mathscr{C}$ still belong to $\mathscr{C}_+$ or $\mathscr{C}$ as functions of these parameters.

More specifically, if $f : X \to \Lambda$ is a morphism and φ is a function in $\mathscr{C}_+(X)$ or $\mathscr{C}(X)$ that is relatively integrable (a notion defined in [7]), one constructs in [7] a function

$$\mu_\Lambda(\varphi) \tag{2.6.1}$$

in $\mathscr{C}_+(\Lambda)$, resp. $\mathscr{C}(\Lambda)$, whose restriction to every fiber of f coincides with the integral of φ restricted to that fiber.

2.7. **The transfer principle.** We are now in the position of explaining how motivic integrals specialize to p-adic integrals and may be used to obtain a general transfer principle allowing to transfer relations between integrals from $\mathbb{Q}_p$ to $\mathbb{F}_p((t))$ and vice-versa.

We shall assume from now on that k is a number field with ring of integers $\mathcal{O}$. We denote by $\mathcal{A}_\mathcal{O}$ the set of p-adic completions of all finite extensions of k and by $\mathcal{B}_\mathcal{O}$ the set of all local fields of characteristic > 0 which are $\mathcal{O}$-algebras.

For K in $\mathscr{C}_\mathcal{O} := \mathcal{A}_\mathcal{O} \cup \mathcal{B}_\mathcal{O}$, we denote by

- R_K the valuation ring
- M_K the maximal ideal
- k_K the residue field
- $q(K)$ the cardinal of k_K
- ϖ_K a uniformizing parameter of R_K.

There exists a unique morphism $\overline{\mathrm{ac}} : K^\times \to k_K^\times$ extending $R_K^\times \to k_K^\times$ and sending ϖ_K to 1. We set $\overline{\mathrm{ac}}(0) = 0$. For $N > 0$, we denote by $\mathcal{A}_{\mathcal{O},N}$ the set of fields K in $\mathcal{A}_\mathcal{O}$ such that k_K has characteristic $> N$, and similarly for $\mathcal{B}_{\mathcal{O},N}$ and $\mathscr{C}_{\mathcal{O},N}$. To be able to interpret our formulas to fields in $\mathscr{C}_\mathcal{O}$, we restrict the language $\mathcal{L}_{\mathrm{DP}}$ to the sub-language $\mathcal{L}_\mathcal{O}$ for which coefficients in the valued field sort are assumed to belong to the subring $\mathcal{O}[[t]]$ of $k((t))$. We denote by $\mathrm{Def}(\mathcal{L}_\mathcal{O})$ the sub-category of Def_k of objects definable in $\mathcal{L}_\mathcal{O}$, and similarly for functions, etc. For instance, for S in $\mathrm{Def}(\mathcal{L}_\mathcal{O})$, we denote by $\mathscr{C}(S, \mathcal{L}_\mathcal{O})$ the ring of constructible functions on S definable in $\mathcal{L}_\mathcal{O}$.

We consider K as a $\mathcal{O}[[t]]$-algebra via

$$\lambda_{\mathcal{O},K} : \sum_{i\in\mathbb{N}} a_i t^i \longmapsto \sum_{i\in\mathbb{N}} a_i \varpi_K^i. \tag{2.7.1}$$

Hence, if we interpret a in $\mathcal{O}[[t]]$ by $\lambda_{\mathcal{O},K}(a)$, every $\mathcal{L}_\mathcal{O}$-formula φ defines for K in $\mathscr{C}_\mathcal{O}$ a subset φ_K of some $K^m \times k_K^n \times \mathbb{Z}^r$. One proves that if two $\mathcal{L}_\mathcal{O}$-formulas φ and φ' define the same subassignment X of $h[m, n, r]$, then $\varphi_K = \varphi'_K$ for K in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$. This allows us to denote by X_K the subset defined by φ_K, for K in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$. Similarly, every $\mathcal{L}_\mathcal{O}$-definable morphism $f : X \to Y$ specializes to $f_K : X_K \to Y_K$ for K in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$.

We now explain how φ in $\mathscr{C}(X, \mathcal{L}_\mathcal{O})$ can be specialized to $\varphi_K : X_K \to \mathbb{Q}$ for K in $\mathscr{C}_{\mathcal{O},N}$ when $N \gg 0$. Let us consider φ in $K_0(\mathrm{RDef}_X(\mathcal{L}_\mathcal{O}))$ of the form $[\pi : W \to X]$ with W in $\mathrm{RDef}_X(\mathcal{L}_\mathcal{O})$. For K in $\mathscr{C}_{\mathcal{O},N}$ with $N \gg 0$, we have $\pi_K : W_K \to X_K$, so we may define $\varphi_K : X_K \to \mathbb{Q}$ by

$$x \longmapsto \mathrm{card}\left(\pi_K^{-1}(x)\right). \tag{2.7.2}$$

For φ in $\mathcal{P}(X)$, we specialize $\mathbb{L}$ into q_K and $\alpha : X \to \mathbb{Z}$ into $\alpha_K : X_K \to \mathbb{Z}$. By tensor product we get $\varphi \mapsto \varphi_K$ for φ in $\mathscr{C}(X, \mathcal{L}_O)$. Note that, under that construction, functions in $\mathscr{C}_+(X, \mathcal{L}_O)$ specialize into non negative functions.

Let K be in $\mathscr{C}_O$ and A be a subset of $K^m \times k_K^n \times \mathbb{Z}^r$. We consider the Zariski closure $\bar{A}$ of the projection of A into $\mathbb{A}_K^m$. One defines a measure μ on A by restriction of the product of the canonical (Serre-Oesterlé) measure on $\bar{A}(K)$ with the counting measure on $k_K^n \times \mathbb{Z}^r$.

Fix a morphism $f : X \to \Lambda$ in $\mathrm{Def}(\mathcal{L}_O)$ and consider φ in $\mathscr{C}(X, \mathcal{L}_O)$. One can show that if φ is relatively integrable, then, for $N \gg 0$, for every K in $\mathscr{C}_{O,N}$, and for every λ in Λ_K, the restriction $\varphi_{K,\lambda}$ of φ_K to $f_K^{-1}(\lambda)$ is integrable.

We denote by $\mu_{\Lambda_K}(\varphi_K)$ the function on Λ_K defined by

$$\lambda \longmapsto \mu(\varphi_{K,\lambda}). \tag{2.7.3}$$

The following theorem says that motivic integrals specialize to the corresponding integrals over local fields of high enough residue field characteristic.

Theorem 2.7.1 (Specialization, Cluckers-Loeser [9] [10])**.** *Let $f : S \to \Lambda$ be a morphism in* $\mathrm{Def}(\mathcal{L}_O)$*. Let φ be in $\mathscr{C}(S, \mathcal{L}_O)$ and relatively integrable with respect to f. For $N \gg 0$, for every K in $\mathscr{C}_{O,N}$, we have*

$$(\mu_\Lambda(\varphi))_K = \mu_{\Lambda_K}(\varphi_K). \tag{2.7.4}$$

We are now ready to state the following abstract transfer principle:

Theorem 2.7.2 (Abstract transfer principle, Cluckers-Loeser [9] [10])**.** *Let φ be in $\mathscr{C}(\Lambda, \mathcal{L}_O)$. There exists N such that for every K_1, K_2 in $\mathscr{C}_{O,N}$ with $k_{K_1} \simeq k_{K_2}$,*

$$\varphi_{K_1} = 0 \quad \textit{if and only if} \quad \varphi_{K_2} = 0. \tag{2.7.5}$$

Putting together the two previous theorems, one immediately gets:

Theorem 2.7.3 (Transfer principle for integrals with parameters, Cluckers-Loeser [9] [10])**.** *Let $S \to \Lambda$ and $S' \to \Lambda$ be morphisms in* $\mathrm{Def}(\mathcal{L}_O)$*. Let φ and φ' be relatively integrable functions in $\mathscr{C}(S, \mathcal{L}_O)$ and $\mathscr{C}(S', \mathcal{L}_O)$, respectively. There exists N such that for every K_1, K_2 in $\mathscr{C}_{O,N}$ with $k_{K_1} \simeq k_{K_2}$,*

$$\mu_{\Lambda_{K_1}}(\varphi_{K_1}) = \mu_{\Lambda_{K_1}}(\varphi'_{K_1}) \quad \textit{if and only if} \quad \mu_{\Lambda_{K_2}}(\varphi_{K_2}) = \mu_{\Lambda_{K_2}}(\varphi'_{K_2}).$$

In the special case where $\Lambda = h[0,0,0]$ and φ and φ' are in $\mathscr{C}(S, \mathcal{L}_O)$ and $\mathscr{C}(S', \mathcal{L}_O)$, respectively, this follows from previous results of Denef-Loeser [12].

Remark 2.7.4. *The previous constructions and statements may be extended directly - with similar proofs - to the global (non affine) setting.*

Note that when $S = S' = \Lambda = h[0,0,0]$, one recovers the classical

Theorem 2.7.5 (Ax-Kochen-Eršov [5] [13])**.** *Let φ be a first order sentence* (*that is, a formula with no free variables*) *in the language of rings. For almost all prime number p, the sentence φ is true in $\mathbb{Q}_p$ if and only if it is true in $\mathbb{F}_p((t))$.*

We define a function $s_M^G(\ell)$ recursively. Assume that $s_M^{G'}$ has been defined for all G' (with Levi subgroup M) such that $\dim G' < \dim G$. Then, set

$$s_M^G(\ell) = J_{M,M}^G(\ell) - \sum_{G' \neq G} \iota_M(G,G') s_M^{G'}(\ell). \tag{9.1.1}$$

The sum runs over $\mathcal{E}_M(G) \setminus \{G\}$. This definition is coherent, because each group $G' \in \mathcal{E}_M(G)$ has M as a Levi subgroup, so that $s_M^{G'}$ is defined.

The conjecture of the weighted fundamental lemma is then that for all G, M, M' as above, we have

$$J_{M,M'}^G(\ell') = \sum_{G'} \iota_{M'}(G,G') s_{M'}^{G'}(\ell'). \tag{9.1.2}$$

for all G-regular elements ℓ' in $\mathbf{c}_{M'}$. The sum on the right runs over $\mathcal{E}_{M'}(G)$.

9.2. Constructibility. By our preceding discussion, we see that the integrand (Equation 8.5.4) of $J_{M,M'}^G(\ell')$ comes as specialization of a constructible function on the definable subassignment

$$Z = \tilde{\mathbf{c}}_H \times_{\tilde{\mathbf{c}}_G} \tilde{\mathbf{g}}_{D,\theta}. \tag{9.2.1}$$

This constructible function depends on parameters a, τ, $\gamma_H \in \tilde{\mathbf{c}}_H$, and $\gamma \in g_{D,\theta,a,\tau}$. If we interpret this p-adically, as we vary the parameter a (under the restriction that it is a unit), the situation specializes to isomorphic groups and Lie algebras. In particular, the fundamental lemma holds for one specialization of a if and only if it holds for all specializations of a. As we vary the generator τ of the Galois group of the unramified field extension F_r/F, we may obtain non-isomorphic data. Different choices of τ correspond to the fundamental lemma for various Lie algebras

$$\mathbf{g}_{D,\theta'} \tag{9.2.2}$$

where θ' and θ generate the same group $\langle\theta'\rangle = \langle\theta\rangle$ of automorphisms of the root data. In particular, for each τ, the constructible version specializes to a version of the p-adic fundamental lemma for Lie algebras.

9.3. The main theorem. We state the transfer principle for the fundamental lemma as two theorems, once in the unweighted case and again in the weighted case. In fact, there is no needed for us to treat these two cases separately; they are both a consequence of the general transfer principle for the motivic integrals of constructible functions given in Theorem 2.7.3. We state them as separate theorems, only because of preprint of Ngô [27], which applies directly to the unweighted case of the fundamental lemma.

Theorem 9.3.1 (Transfer Principle for the Fundamental Lemma)**.** *Let (D,θ) be given. Suppose that the fundamental lemma holds for all p-adic fields of positive characteristic for the endoscopic groups attached to (D,θ'), as θ' ranges over automorphisms of the root data such that $\langle\theta'\rangle = \langle\theta\rangle$. Then, the fundamental lemma holds for all p-adic fields of characteristic zero with sufficiently large residual characteristic p (in the same context of all endoscopic groups attached to (D,θ')).*

Theorem 9.3.2 (Transfer Principle for the weighted Fundamental Lemma)**.** *Let* (D,θ) *be given. Suppose that the weighted fundamental lemma holds for all p-adic fields of positive characteristic for the endoscopic groups attached to* (D,θ')*, as* θ' *ranges over automorphisms of the root data such that* $\langle\theta'\rangle = \langle\theta\rangle$*. Then, the weighted fundamental lemma holds for all p-adic fields of characteristic zero with sufficiently large residual characteristic p (in the same context of all endoscopic groups attached to* (D,θ')*).*

Proof. We have successfully represented all the data entering into the fundamental lemma within the general framework of identities of motivic integrals of constructible functions. By the transfer principle given in Theorem 2.7.3, the fundamental lemma holds for all p-adic fields of characteristic zero, for sufficiently large primes p. □

By the main result of [16], the unweighted fundamental lemma holds for all elements of the Hecke algebra for all p, once it holds for all sufficiently large p (for a collection of endoscopic data obtained by descent from the original data (D,θ)). Thus, in the unweighted situation, we can derive the fundamental lemma for all local fields of characteristic zero, without restriction on p, once the fundamental lemma is known for a suitable collection of cases in positive characteristic.

10. A

10.1. Adding exponentials. It is also possible to enlarge $\mathscr{C}(X)$ to a ring $\mathscr{C}(X)^{\mathrm{exp}}$ also containing motivic analogues of exponential functions and to construct a natural extension of the previous theory to $\mathscr{C}^{\mathrm{exp}}$.

This is performed as follows in [9] [10]. Let X be in Def_k. We consider the category $\mathrm{RDef}_X^{\mathrm{exp}}$ whose objects are triples $(Y \to X, \xi, g)$ with Y in RDef_X and $\xi : Y \to h[0,1,0]$ and $g : Y \to h[1,0,0]$ morphisms in Def_k. A morphism $(Y' \to X, \xi', g') \to (Y \to X, \xi, g)$ in $\mathrm{RDef}_X^{\mathrm{exp}}$ is a morphism $h : Y' \to Y$ in Def_X such that $\xi' = \xi \circ h$ and $g' = g \circ h$. The functor sending Y in RDef_X to $(Y, 0, 0)$, with 0 denoting the constant morphism with value 0 in $h[0,1,0]$, resp. $h[1,0,0]$ being fully faithful, we may consider RDef_X as a full subcategory of $\mathrm{RDef}_X^{\mathrm{exp}}$. To the category $\mathrm{RDef}_X^{\mathrm{exp}}$ one assigns a Grothendieck ring $K_0(\mathrm{RDef}_X^{\mathrm{exp}})$ defined as follows. As an abelian group it is the quotient of the free abelian group over symbols $[Y \to X, \xi, g]$ with $(Y \to X, \xi, g)$ in $\mathrm{RDef}_X^{\mathrm{exp}}$ by the following four relations

$$[Y \to X, \xi, g] = [Y' \to X, \xi', g'] \tag{10.1.1}$$

for $(Y \to X, \xi, g)$ isomorphic to $(Y' \to X, \xi', g')$,

$$\begin{aligned}&[(Y \cup Y') \to X, \xi, g] + [(Y \cap Y') \to X, \xi_{|Y\cap Y'}, g_{|Y\cap Y'}]\\ &\qquad = [Y \to X, \xi_{|Y}, g_{|Y}] + [Y' \to X, \xi_{|Y'}, g_{|Y'}]\end{aligned} \tag{10.1.2}$$

for Y and Y' definable subassignments of some W in RDef_X and ξ, g defined on $Y \cup Y'$,

$$[Y \to X, \xi, g + h] = [Y \to X, \xi + \overline{h}, g] \tag{10.1.3}$$