Agenda

- Consider an approximation of $u \in L^2_\omega(I)$ in terms of a TRUNCATED CHEBYSHEV SERIES $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$
- Cancel the projections of the residual $R_N = u - u_N$ on the $N+1$ first basis function (i.e., the Chebyshev polynomials)

$$(R_N, T_l)_\omega = \int_{-1}^{1} \left(u T_l \omega - \sum_{k=0}^{N} \hat{u}_k T_k T_l \omega \right) dx = 0, \quad l = 0, \dots, N$$

- Taking into account the orthogonality condition, expressions for the Chebyshev expansions coefficients are obtained

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k \omega \, dx,$$

which can be evaluated using, e.g., the GAUSS-LOBATTO-CHEBYSHEV QUADRATURES .
- QUESTION — What happens on the boundary?

Theorem

Let $P_N : L^2_\omega(I) \to \mathbb{P}_N$ be the orthogonal projection on the subspace $\mathbb{P}_N$ of polynomials of degree $\le N$. For all μ and σ such that $0 \le \mu \le \sigma$, there exists a constant C such that

$$\|u - P_N u\|_{\mu,\omega} < C N^{e(\mu,\sigma)} \|u\|_{\sigma,\omega}$$

$$\text{where} \qquad e(\mu,\sigma) = \begin{cases} 2\mu - \sigma - \dfrac{1}{2} & \text{for } \mu > 1, \\ \dfrac{3}{2}\mu - \sigma & \text{for } 0 \le \mu \le 1 \end{cases}$$

"Philosophy" of the proof.

1. First establish continuity of the mapping $u \to \tilde{u}$, where $\tilde{u}(\theta) = u(\cos(\theta))$, from the weighted Sobolev space $H^m_\omega(I)$ into the corresponding periodic Sobolev space $H^m_p(-\pi,\pi)$
2. Then leverage analogous approximation error bounds established for the case of trigonometric basis functions

- Consider an approximation of $u \in L^2_\omega(I)$ in terms of a truncated Chebyshev series (expansion coefficients as the unknowns) $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$
- Cancel the residual $R_N = u - u_N$ on the set of Gauss-Lobatto-Chebyshev collocation points x_j, $j = 0, \dots, N$ (one could choose other sets of collocation points as well)

$$u(x_j) = \sum_{k=0}^{N} \hat{u}_k T_k(x_j), \quad j = 0, \dots, N$$

- Noting that $T_k(x_j) = \cos\left(k \cos^{-1}(\cos(\frac{j\pi}{N}))\right) = \cos(k\frac{j\pi}{N})$ and denoting $u_j \triangleq u(x_j)$ we obtain

$$u_j = \sum_{k=0}^{N} \hat{u}_k \cos\left(k\frac{\pi j}{N}\right), \quad j = 0, \dots, N$$

- The above system of equations can be written as $U = \mathcal{T}\hat{U}$, where U and $\hat{U}$ are vectors of grid values and expansion coefficients, respectively.

- In fact, the matrix $\mathcal{T}$ is invertible and

$$[\mathcal{T}^{-1}]_{jk} = \frac{2}{\overline{c}_j \overline{c}_k N} \cos\left(\frac{k\pi j}{N}\right), \qquad j,k = 0,\dots,N$$

- Consequently, the expansion coefficients can be expressed as follows

$$\hat{u}_k = \frac{2}{\overline{c}_k N} \sum_{j=0}^{N} \frac{1}{\overline{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right) = \frac{2}{\overline{c}_k N} \sum_{j=0}^{N} \frac{1}{\overline{c}_j} u_j \Re\left[e^{i\left(\frac{k\pi j}{N}\right)}\right], \qquad k = 0,\dots,N$$

  Note that this expression is nothing else than the COSINE TRANSFORM of U which can be very efficiently evaluated using a COSINE FFT

- The same expression can be obtained by
  - multiplying each side of $u_j = \sum_{k=0}^{N} \hat{u}_k T_k(x_j)$ by $\frac{T_l(x_j)}{\overline{c}_j}$
  - summing the resulting expression from $j = 0$ to $j = N$
  - using the DISCRETE ORTHOGONALITY RELATION

$$\frac{\pi}{N} \sum_{j=0}^{N} \frac{1}{\overline{c}_j} T_k(\tilde{\xi}_j) T_l(\tilde{\xi}_j) = \frac{\pi \overline{c}_k}{2} \delta_{kl}$$

- Note that the expression for the Discrete Chebyshev Transform

$$\hat{u}_k = \frac{2}{\bar{c}_k N} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} u_j \cos\left(\frac{k\pi j}{N}\right), \quad k = 0, \dots, N$$

can also be obtained by using the Gauss-Lobatto-Chebyshev quadrature to approximate the continuous expressions

$$\hat{u}_k = \frac{2}{\pi c_k} \int_{-1}^{1} u T_k \omega \, dx, \quad k = 0, \dots, N,$$

Such an approximation is EXACT for $u \in \mathbb{P}_N$

- Analogous expressions for the Discrete Chebyshev Transforms can be derived for other set of collocation points (Gauss, Gauss-Radau)
- Note similarities with respect to the case periodic functions and the Discrete Fourier Transform

- As was the case with Fourier spectral methods, there is a very close connection between COLLOCATION-BASED INTERPOLATION and GALERKIN APPROXIMATION
- DISCRETE CHEBYSHEV TRANSFORM can be associated with an INTERPOLATION OPERATOR $P_C : C^0(I) \to \mathbb{R}^N$ defined such that $(P_C u)(x_j) = u(x_j)$, $j = 0, \dots, N$ (where x_j are the Gauss-Lobatto collocation points)

Theorem

Let $s > \frac{1}{2}$ and σ be given and $0 \le \sigma \le s$. There exists a constant C such that

$$\|u - P_C u\|_{\sigma,\omega} < C N^{2\sigma - s} \|u\|_{s,\omega}$$

for all $u \in H^s_\omega(I)$.

Outline of the Proof.

Changing the variables to $\tilde{u}(\theta) = u(\cos(\theta))$ we convert this problem to a problem already analyzed in the context of the Fourier interpolation for periodic functions $\square$

- Relation between the GALERKIN and COLLOCATION coefficients, i.e.,

$$\hat{u}_k^e = \frac{2}{\pi c_k}\int_{-1}^{1} u(x)\,T_k(x)\omega(x)\,dx, \qquad k = 0,\dots,N$$

$$\hat{u}_k^c = \frac{2}{\bar{c}_k N}\sum_{j=0}^{N}\frac{1}{\bar{c}_j}u_j\cos\left(\frac{k\pi j}{N}\right), \qquad k = 0,\dots,N$$

- Using the representation $u(x) = \sum_{l=0}^{\infty}\hat{u}_l^e\,T_l(x)$ in the latter expression and invoking the discrete orthogonality relation we obtain

$$\begin{aligned}\hat{u}_k^c &= \frac{2}{\bar{c}_k N}\sum_{l=0}^{N}\hat{u}_l^e\left[\sum_{j=0}^{N}\frac{1}{\bar{c}_j}T_k(x_j)\,T_l(x_j)\right] + \frac{2}{\bar{c}_k N}\sum_{l=N+1}^{\infty}\hat{u}_l^e\left[\sum_{j=0}^{N}\frac{1}{\bar{c}_j}T_k(x_j)\,T_l(x_j)\right], \\ &= \hat{u}_k^e \qquad\qquad + \frac{2}{\bar{c}_k N}\sum_{l=N+1}^{\infty}\hat{u}_l^e\,C_{kl}\end{aligned}$$

$$\text{where}\quad C_{kl} = \sum_{j=0}^{N}\frac{1}{\bar{c}_j}T_k(x_j)\,T_l(x_j) = \sum_{j=0}^{N}\frac{1}{\bar{c}_j}\cos\left(\frac{kj\pi}{N}\right)\cos\left(\frac{lj\pi}{N}\right)$$

$$= \frac{1}{2}\sum_{j=0}^{N}\frac{1}{\bar{c}_j}\left[\cos\left(\frac{k-l}{N}j\pi\right) + \cos\left(\frac{k+l}{N}j\pi\right)\right]$$

- Using the identity

$$\sum_{j=0}^{N} \cos\left(\frac{pi\pi}{N}\right) = \begin{cases} N+1, & \text{if } p = 2mN, \;\; m = 0, \pm 1, \pm 2, \dots \\ \frac{1}{2}[1 + (-1)^p] & \text{otherwise} \end{cases}$$

we can calculate C_{kl} which allows us to express the relation between the Galerkin and collocation coefficients as follows

$$\hat{u}_k^c = \hat{u}_k^e + \frac{1}{\overline{c}_k}\left[\sum_{\substack{m=1 \\ 2mN > N-k}}^{\infty} \hat{u}^e_{k+2mN} + \sum_{\substack{m=1 \\ 2mN > N+k}}^{\infty} \hat{u}^e_{-k+2mN}\right]$$

- The terms in square brackets represent the ALIASING ERRORS . Their origin is precisely the same as in the Fourier (pseudo)-spectral method.
- Aliasing errors can be removed using the 3/2 APPROACH in the same way as in the Fourier (pseudo)-spectral method

- expressing the first N Chebyshev polynomials as functions of x^k, $k = 1, \dots, N$

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

which can be written as $V = \mathbb{K}X$, where $[V]_k = T_k(x)$, $[X]_k = x^k$, and $\mathbb{K}$ is a LOWER-TRIANGULAR matrix

- Solving this system (trivially!) results in the following RECIPROCAL RELATIONS

$$\begin{aligned} 1 &= T_0(x), \\ x &= T_1(x), \\ x^2 &= \frac{1}{2}[T_0(x) + T_2(x)], \\ x^3 &= \frac{1}{4}[3T_1(x) + T_3(x)], \\ x^4 &= \frac{1}{8}[3T_0(x) + 4T_2(x) + T_4(x)] \end{aligned}$$

- Find the best polynomial approximation of order 3 of $f(x) = e^x$ on $[-1, 1]$
- Construct the (Maclaurin) expansion

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

- Rewrite the expansion in terms of Chebyshev polynomials using the reciprocal relations

$$e^x = \frac{81}{64}T_0(x) + \frac{9}{8}T_1(x) + \frac{13}{48}T_2(x) + \frac{1}{24}T_3(x) + \frac{1}{192}T_4(x) + \dots$$

- Truncate this expansion to the 3^{rd} order and translate the expansion back to the x^k representation
- Truncation error is given by the magnitude of the first truncated term; Note that the Chebyshev Expansion coefficients are much smaller than the corresponding Taylor expansion coefficients !
- How is it possible – the same number of expansion terms, but higher accuracy?

- Assume function approximation in the form $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$
- First, note that CHEBYSHEV PROJECTION and DIFFERENTIATION do not commute, i.e., $P_N(\frac{du}{dx}) \neq \frac{d}{dx}(P_N u)$
- Sequentially applying the recurrence relation $2T_k = \frac{T'_{k+1}}{k+1} - \frac{T'_{k-1}}{k-1}$ we obtain

$$T'_k(x) = 2k \sum_{p=0}^{K} \frac{1}{c_{k-1-2p}} T_{k-1-2p}(x), \quad \text{where } K = \left[\frac{k-1}{2}\right]$$

- Consider the first derivative

$$u'_N(x) = \sum_{k=0}^{N} \hat{u}_k T'_k(x) = \sum_{k=0}^{N} \hat{u}_k^{(1)} T_k(x)$$

where, using the above expression for $T'_k(x)$, we obtain the expansion coefficients as

$$\hat{u}_k^{(1)} = \frac{2}{c_k} \sum_{\substack{p=k+1 \\ (p+k)\,\text{odd}}}^{N} p\hat{u}_p, \quad k = 0, \dots, N-1, \quad \text{and} \quad \hat{u}_N^{(1)} = 0$$

- Spectral differentiation (with the expansion coefficients as unknowns) can thus be written as
$$\hat{U}^{(1)} = \hat{\mathbb{D}}\hat{U},$$
where $\hat{U} = [\hat{u}_0 \ \dots, \hat{u}_N]^T$, $\hat{U}^{(1)} = [\hat{u}_0^{(1)} \ \dots, \hat{u}_N^{(1)}]^T$, and $\hat{\mathbb{D}}$ is an UPPER-TRIANGULAR matrix with entries deduced based on the previous expression
- For the second derivative one obtains similarly
$$u_N''(x) = \sum_{k=0}^{N} \hat{u}_k^{(2)} T_k(x)$$
$$\hat{u}_k^{(2)} = \frac{1}{c_k} \sum_{\substack{p=k+2 \\ (p+k)\text{ even}}}^{N} p(p^2-k^2)\hat{u}_p, \quad k = 0, \dots, N-2$$
and $\hat{u}_N^{(2)} = \hat{u}_{N-1}^{(2)} = 0$
- QUESTION — What is the structure of the second-order differentiation matrix?

- Assume the function $u(x)$ is approximated in terms of its nodal values, i.e.,

$$u(x) \cong u_N(x) = \sum_{j=0}^{N} u(x_j) C_j(x),$$

where $\{x_j\}$ are the Gauss-Lobatto-Chebyshev points and $C_j(x)$ are the associated Cardinal Functions

$$C_j(x) = (-1)^{j+1} \frac{(1-x^2)}{c_j N^2 (x - x_j)} \frac{dT_N(x)}{dx} = \frac{2}{N p_j} \sum_{m=0}^{N} \frac{1}{p_m} T_m(x_j) T_m(x),$$

where

$$p_j = \begin{cases} 2 & \text{for } j = 0, N, \\ 1 & \text{for } j = 1, \dots, N-1 \end{cases}, \qquad c_j = \begin{cases} 2 & \text{for } j = N, \\ 1 & \text{for } j = 0, \dots, N-1 \end{cases}$$

- The Differentiation Matrix $\mathbb{D}^{(p)}$ relating the nodal values of the p-th derivative $u_N^{(p)}$ to the nodal values of u is obtained by differentiating the cardinal function appropriate number of times

$$u_N^{(p)}(x_j) = \sum_{k=0}^{N} \frac{d^{(p)} C_k(x_j)}{dx^{(p)}} u(x_k) = \sum_{k=0}^{N} d_{jk}^{(p)} u(x_k), \quad j = 0, \dots, N$$

- Expressions for the entries of the DIFFERENTIATION MATRIX $d_{jk}^{(1)}$ at the the GAUSS-LOBATTO-CHEBYSHEV collocation points

$$
\begin{aligned}
d_{jk}^{(1)} &= \frac{\overline{c}_j}{\overline{c}_k}\frac{(-1)^{j+k}}{x_j - x_k}, \qquad && 0 \le j,k \le N,\ j \neq k, \\
d_{jj}^{(1)} &= -\frac{x_j}{2(1-x_j^2)}, \qquad && 1 \le j \le N-1, \\
d_{00}^{(1)} &= -d_{NN}^{(1)} = \frac{2N^2+1}{6}, &&
\end{aligned}
$$

- Thus in the matrix (operator) notation

$$U^{(1)} = \mathbb{D}U$$

- Note that ROWS of the differentiation matrix $\mathbb{D}$ are in fact equivalent to N-th order asymmetric finite-difference formulas on a nonuniform grid; in other words, spectral differentiation using nodal values as unknowns is equivalent to finite differences employing ALL N GRID POINTS AVAILABLE

- Expressions for the entries of SECOND-ORDER DIFFERENTIATION MATRIX $d_{jk}^{(2)}$ at the the GAUSS-LOBATTO-CHEBYSHEV collocation points ($U^{(2)} = \mathbb{D}^{(2)}U$)

$$d_{jk}^{(2)} = \frac{(-1)^{j+k}}{\overline{c}_k}\frac{x_j^2 + x_j x_k - 2}{(1-x_j^2)(x_j-x_k)^2}, \qquad 1 \le j \le N-1,\ 0 \le k \le N,\ j \ne k$$

$$d_{jj}^{(2)} = -\frac{(N^2-1)(1-x_j^2)+3}{3(1-x_j^2)^2}, \qquad 1 \le j \le N-1,$$

$$d_{0k}^{(2)} = \frac{2}{3}\frac{(-1)^k}{\overline{c}_k}\frac{(2N^2+1)(1-x_k)-6}{(1-x_k)^2}, \qquad 1 \le k \le N$$

$$d_{Nk}^{(2)} = \frac{2}{3}\frac{(-1)^{N+k}}{\overline{c}_k}\frac{(2N^2+1)(1+x_k)-6}{(1+x_k)^2} \qquad ,0 \le k \le N-1$$

$$d_{00}^{(2)} = d_{NN}^{(2)} = \frac{N^4-1}{15},$$

- Note that $d_{jk}^{(2)} = \sum_{p=0}^{N} d_{jp}^{(1)} d_{pk}^{(1)}$
- Interestingly, $\mathbb{D}^2$ is not a SYMMETRIC MATRIX ...

- Consider an ELLIPTIC BOUNDARY VALUE PROBLEM (BVP) :

$$\begin{aligned} -\nu u'' + au' + bu &= f, && \text{in } [-1,1] \\ \alpha_- u + \beta_- u' &= g_- && x = -1 \\ \alpha_+ u + \beta_+ u' &= g_+ && x = 1 \end{aligned}$$

- Chebyshev polynomials do not satisfy homogeneous boundary conditions, hence standard Galerkin approach is not directly applicable.
- BASIS RECOMBINATION :
  - Convert the BVP to the corresponding form with HOMOGENEOUS BOUNDARY CONDITIONS
  - Take linear combinations of Chebyshev polynomials to construct a new basis satisfying HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS $\varphi_k(\pm 1) = 0$

$$\varphi_k(x) = \begin{cases} T_k(x) - T_0(x) = T_k - 1, & k - \text{even} \\ T_k(x) - T_1(x), & k - \text{odd} \end{cases}$$

Note that the new basis preserves orthogonality

- THE TAU METHOD (Lanczos, 1938) consists in using a Galerkin approach in which explicit enforcement of the boundary conditions replaces projections on some of the test functions
- Consider the residual

$$R_N(x) = -\nu u_N'' + a u_N' + b u_N - f,$$

where $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

- Cancel projections of the residual on the first $N-2$ basis functions

$$(R_N, T_l)_\omega = \sum_{k=0}^{N} \left(-\nu \hat{u}_k^{(2)} + a\hat{u}_k^{(1)} + b\hat{u}_k\right) \int_{-1}^{1} T_k T_l \omega \, dx - \int_{-1}^{1} f T_l \omega \, dx, \quad l = 0, \dots, N-2$$

- Thus, using orthogonality, we obtain

$$-\nu \hat{u}_k^{(2)} + a\hat{u}_k^{(1)} + b\hat{u}_k = \hat{f}_k, \quad k = 0, \dots, N-2$$

where $\hat{f}_k = \int_{-1}^{1} f \, T_k \, \omega \, dx$

- Noting that $T_k(\pm 1) = (\pm 1)^k$ and $T_k'(\pm 1) = (\pm 1)^{k+1} k^2$, the BOUNDARY CONDITIONS are enforced by supplementing the residual equations with

$$\sum_{k=0}^{N} (-1)^k (\alpha_- - \beta_- k^2) \hat{u}_k = g_-$$

$$\sum_{k=0}^{N} (-1)^k (\alpha_+ + \beta_+ k^2) \hat{u}_k = g_+$$

- Expressing $\hat{u}_k^{(2)}$ and $\hat{u}_k^{(1)}$ in terms of $\hat{u}_k$ via the Chebyshev spectral differentiation matrices we obtain the following system

$$\mathbb{A} \hat{U} = \hat{F}$$

where $\hat{U} = [\hat{u}_0, \dots, \hat{u}_N]^T$, $F = [\hat{f}_0, \dots, \hat{f}_{N-2}, g_-, g_+]$ and the matrix $\mathbb{A}$ is obtained by adding the two rows representing the boundary conditions (see above) to the matrix $\mathbb{A}_1 = -\nu \hat{\mathbb{D}}^2 + a\hat{\mathbb{D}} + bI$.

- When the domain boundary is not just a point (e.g., in 2D / 3D), formulation of the Tau method becomes somewhat more involved

- Consider the residual

$$R_N(x) = -\nu u_N'' + a u_N' + b u_N - f,$$

where $u_N(x) = \sum_{k=0}^{N} \hat{u}_k T_k(x)$

- Cancel this residual at $N-1$ Gauss-Lobatto-Chebyshev collocation points located in the interior of the domain

$$-\nu u_N''(x_j) + a u_N'(x_j) + b u_N(x_j) = f(x_j), \quad j = 1, \dots, N-1$$

- Enforce the two boundary conditions at endpoints

$$\alpha_- u_N(x_N) + \beta_- u_N'(x_N) = g_-$$
$$\alpha_+ u_N(x_0) + \beta_+ u_N'(x_0) = g_-$$

Note that this shows the utility of using the Gauss-Lobatto-Chebyshev collocation points

- Consequently, the following system of $N+1$ equations is obtained

$$\sum_{k=0}^{N}(-\nu d_{jk}^{(2)} + a d_{jk}^{(1)}) u_N(x_j) + b u_N(x_j) = f(x_j), \quad j = 1, \dots, N-1$$

$$\alpha_- u_N(x_N) + \beta_- \sum_{k=0}^{N} d_{Nk}^{(1)} u_N(x_k) = g_-$$

$$\alpha_+ u_N(x_0) + \beta_+ \sum_{k=0}^{N} d_{0k}^{(1)} u_N(x_k) = g_+$$

which can be written as $\mathbb{A}_c U = F$, where $[\mathbb{A}_c]_{jk} = [\mathbb{A}_{c0}]_{jk}$, $j, k = 1, \dots, N-1$ with $\mathbb{A}_{c0}$ given by

$$\mathbb{A}_{c0} = (-\nu \mathbb{D}^2 + a\mathbb{D} + b\mathbb{I})U$$

and the BOUNDARY CONDITIONS above added as the rows 0 and N of $\mathbb{A}_c$

- Note that the matrix corresponding to this system of equations may be POORLY CONDITIONED , so special care must be exercised when solving this system for large N.

- Similar approach can be used when the nodal values $u(x_j)$, rather than the Chebyshev coefficients $\hat{u}_k$ are unknowns

- When the equation has NONCONSTANT COEFFICIENTS , similar difficulties as in the Fourier case are encountered (evaluation of CONVOLUTION SUMS)
- Consequently, the COLLOCATION (pseudo-spectral) approach is preferable along the guidelines laid out in the case of the Fourier spectral methods
- Assuming $a = a(x)$ in the elliptic boundary value problem, we need to make the following modification to $\mathbb{A}_c$:

$$\mathbb{A}'_{c0} = (-\nu\mathbb{D}^2 + \mathbb{D}' + b\mathbb{I})U,$$

  where $\mathbb{D}' = [a(x_j)d^{(1)}_{jk}]$, $j, k = 1, \dots, N$
- For the Burgers equation $\partial_t u + \frac{1}{2}\partial_x u^2 - \nu\partial_x^2 u$ we obtain at every time step n

$$(\mathbb{I} - \Delta t\,\nu\,\mathbb{D}^{(2)})U^{n+1} = U^n - \frac{1}{2}\Delta t\,\mathbb{D}\,W^n,$$

  where $[W^n]_j = [U^n]_j[U^n]_j$; Note that an algebraic system has to be solved at each time step

Epilogue — Domain Decomposition

- Motivation:
  - treatment of problem in IRREGULAR DOMAINS
  - STIFF PROBLEMS
- PHILOSOPHY — partition the original domain Ω into a number of SUBDOMAINS $\{\Omega_m\}_{m=1}^{M}$ and solve the problem separately on each those while respecting consistency conditions on the interfaces
- Spectral Element Method
  - consider a collection of problems posed on each subdomain Ω_m

$$\mathcal{L}u_m = f$$

$$u_{m-1}(a_m) = u_m(a_m), \qquad u_m(a_{m+1}) = u_{m+1}(a_{m+1})$$

  - Transform each subdomain Ω_m to $I = [-1, 1]$
  - use a separate set of N_m ORTHOGONAL POLYNOMIALS to approximate the solution on every subinterval
  - boundary conditions on interfaces provide coupling between problems on subdomains