PART V

Wavelets & Multiresolution Analysis

WAVELETS — OVERVIEW (I)

- What is wrong with Fourier analysis ???
  - All spatial information is hidden in the phases of the expansion coefficients and therefore not readily available
  - Localized functions ("bumps") tend to have a very complex representation in Fourier space
  - Local modification of the function affects its whole Fourier transform
  - If the dominant frequency changes in space, only average frequencies are encoded in Fourier coefficients
- Remedy — need an analysis tool that will encode both space (time) and frequency information at the same time
- Following the convention, will work with time (t) and frequency (ω)

WAVELETS — OVERVIEW (II)

- From Discrete Fourier Transform to Integral Fourier Transform — Consider the space $L_2(\mathbb{R})$ of square–integrable functions defined on $\mathbb{R}$; if $f \in L_2(\mathbb{R})$ satisfies suitable decay conditions at $\pm\infty$ (which??), the Discrete Fourier Transform can be replaced with the Integral Fourier Transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega t}\,dt$$

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega$$

- Interestingly, the Fourier Transforms (both discrete and integral) are constructed as "superpositions" of dilations of the function $w(x) = e^{ix}$ $(w_k(t) = w(kt))$
- Want to construct an integral transform using a basis function ψ which is very localized (a "wavelet"); we will therefore need:
  - dilations
  - translations

WAVELETS — GABOR TRANSFORM (I)

- The history begins with a windowed Fourier transform known as the Gabor Transform (1946)

$$(\mathcal{G}_b^{\alpha} f)(\omega) = \int_{-\infty}^{\infty} \left(f(t)e^{-i\omega t}\right) g_{\alpha}(t-b)\,dt,$$

  where the window function is given by $g_{\alpha}(t) = \frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{t^2}{4\alpha}}$ with $\alpha > 0$
- Note that the Fourier transform if a Gaussian function is another Gaussian function, i.e., $\int_{-\infty}^{\infty} e^{-i\omega x}e^{ax^2}\,dx = \sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$
- Note also that the window function has the following normalization $\int_{-\infty}^{\infty} g_{\alpha}(t-b)\,db = \int_{-\infty}^{\infty} g_{\alpha}(x)\,dx = 1$
- Therefore, for the Gabor transform we obtain

$$\int_{-\infty}^{\infty} (\mathcal{G}_b^{\alpha} f)(\omega)\,db = \hat{f}(\omega), \quad \omega \in \mathbb{R}$$

- Thus, the set $\{\mathcal{G}_b^{\alpha} f : b \in \mathbb{R}\}$ of Gabor transforms of f decomposes the Fourier transforms $\hat{f}$ of f exactly to give its local spectral information

Wavelets — Gabor Transform (II)

- The width of the window function can be characterized by employing the notion of the standard deviation

$$\Delta_{g_\alpha} \triangleq \frac{1}{\|g_\alpha\|_2} \left\{ \int_{-\infty}^{\infty} x^2 g_\alpha^2(x)\, dx \right\}^{1/2}$$

- Note that for $\alpha > 0$ $\Delta_{g_\alpha} = \sqrt{\alpha}$
  Proof:
  - $\|g_\alpha\| = (8\pi\alpha)^{-1/4}$ can be evaluated setting $\omega = 0$ and $a = (2\alpha)^{-1}$ in the expression for the Fourier transform of a Gaussian function
  - $\int_{-\infty}^{\infty} x^2 g_\alpha^2(x)\, dx$ can be evaluated twice differentiating the Fourier transform of a Gaussian function and again setting $\omega = 0$ and $a = (2\alpha)^{-1}$
- Instead of localizing the Fourier transform of f, the Gabor transform may equivalently be regarded as windowing f with the window function $\mathcal{G}_{b,\omega}^{\alpha}$

$$(\mathcal{G}_b^\alpha f)(\omega) = (f, \mathcal{G}_{b,\omega}^\alpha) = \int_{-\infty}^{\infty} f(t)\overline{\mathcal{G}_{b,\omega}^\alpha(t)}\, dt$$

Wavelets — Gabor Transform (III)

- Using the Parseval identity and noting that

$$\hat{\mathcal{G}}_{b,\omega}^\alpha(\eta) = e^{-ib(\eta-\omega)} e^{-\alpha(\eta-\omega)^2}$$

we obtain for the Gabor transform

$$\begin{aligned}(\mathcal{G}_b^\alpha f)(\omega) = (f, \mathcal{G}_{b,\omega}^\alpha) &= \frac{1}{2\pi}(\hat{f}, \hat{\mathcal{G}}_{b,\omega}^\alpha) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\eta) e^{ib(\eta-\omega)} e^{-\alpha(\eta-\omega)^2}\, d\eta \\ &= \frac{e^{-ib\omega}}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} \left(e^{ib\eta}\hat{f}(\eta)\right) g_{1/4\alpha}(\eta-\omega)\, d\eta \\ &= \frac{e^{-ib\omega}}{2\sqrt{\pi\alpha}} (\mathcal{G}_\omega^{1/4\alpha}\hat{f})(-b)\end{aligned}$$

- The third line (in red) indicates that up to a multiplicative factor $\sqrt{\frac{\pi}{\alpha}}e^{-ib\omega}$
  - the windowed Fourier transform of f with g_α at $t = b$,
  - the window inverse Fourier transform of $\hat{f}$ with $g_{1/4\alpha}$ at $\eta = \omega$

  are equal!

Wavelets — Uncertainty principle (I)

- Consider more general window functions $w \in L_2(\mathbb{R})$ which satisfy the requirement

$$tw(t) \in L_2(\mathbb{R})$$

  It can be shown that
  - $|t|^{1/2} w(t) \in L_2(\mathbb{R})$
  - $w \in L_1(\mathbb{R})$
  - the Fourier transform $\hat{w}$ is continuous
  - $\hat{w} \in L_2(\mathbb{R})$

  Note, however, that in general $x\hat{w}(x) \notin L_2(\mathbb{R})$, therefore w may not in general be a frequency window function
- If $w \in L_2(\mathbb{R})$ is chosen so that both w and $\hat{w}$ satisfy the above condition, then the window Fourier transform

$$(\tilde{\mathcal{G}}_b f)(\omega) = \int_{-\infty}^{\infty} \left(f(t)e^{-i\omega t}\right)\overline{w(t-b)}\, dt = (f, W_{b,\omega}),$$

  where $W_{b,\omega} = e^{i\omega t} w(t-b)$, is called a short–time Fourier transform

Wavelets — Uncertainty principle (II)

- We can define the center x^* and radius Δ_w of w as

$$x^* \triangleq \frac{1}{\|w\|_2^2}\int_{-\infty}^{\infty} t|w(t)|^2\, dt, \qquad \Delta_w \triangleq \frac{1}{\|w\|_2}\left\{\int_{-\infty}^{\infty}(t-x^*)^2|w(t)|^2\, dt\right\}^{1/2}$$

- Then, $(\tilde{\mathcal{G}}_b f)(\omega)$ gives local information on f in the time–window

$$[x^* + b - \Delta_w, x^* + b + \Delta_w]$$

- We can determine the center ω^* and the radius $\Delta_{\hat{w}}$ of the (frequency) window function $\hat{w}$ using formulae similar to the above
- Defining $V_{b,\omega}(\eta) \triangleq \frac{1}{2\pi}\hat{W}_{b,\omega}(\eta) = \frac{1}{2\pi}e^{ib\omega}e^{-ib\eta}\hat{w}(\eta-\omega)$, which is also a window function with the center $\omega^* + \omega$ and radius $\Delta_{\hat{w}}$ we can write (using the Parseval identity)

$$(\tilde{\mathcal{G}}_b f)(\omega) = (f, W_{b,\omega}) = (\hat{f}, V_{b,\omega})$$

- Thus, $(\tilde{\mathcal{G}}_b f)(\omega)$ also gives local spectral information about t in the frequency window

$$[\omega^* + \omega - \Delta_{\hat{w}}, \omega^* + \omega + \Delta_{\hat{w}}]$$

Wavelets — Uncertainty principle (III)

- Therefore by choosing $w \in L_2(\mathbb{R})$ such that both $xw(x) \in L_2(\mathbb{R})$ and $x\hat{w}(x) \in L_2(\mathbb{R})$ to define a windowed Fourier transform $(\tilde{\mathcal{G}}_b f)(\omega)$ we obtain localization in a time–frequency window

$$[x^* + b - \Delta_w, x^* + b + \Delta_w] \times [\omega^* + \omega - \Delta_{\hat{w}}, \omega^* + \omega + \Delta_{\hat{w}}]$$

  with area equal to $4\Delta_w \Delta_{\hat{w}}$

- In fact, there is a relation between possible time and frequency windows which is made precise in the following theorem
- Heisenberg Uncertainty Principle — Let $w \in L_2(\mathbb{R})$ be chosen so that $xw(x) \in L_2(\mathbb{R})$ and $x\hat{w}(x) \in L_2(\mathbb{R})$. Then

$$\Delta_w \Delta_{\hat{w}} \geq \frac{1}{2}$$

  Furthermore, equality is attained if and only iff

$$w(t) = ce^{i\alpha t} g_\alpha(t-b),$$

  where $c \neq 0$, $\alpha > 0$, and $a, b \in \mathbb{R}$.

Wavelets — Uncertainty principle (IV)

- Proof of the Heisenberg Uncertainty Principle
  - Let us assume that the centers x^* and ω^* are zero (if they are not, then we can modify w as $\tilde{w}(t) = e^{-i\omega^* t} f(t + x^*)$)
  - We observe that

$$\Delta_w^2 \Delta_{\hat{w}}^2 = \frac{\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt \int_{-\infty}^{\infty} \omega^2 |\hat{w}(\omega)|^2 \, d\omega}{\|w\|_2^2 \|\hat{w}\|_2^2}$$
$$= \frac{\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt \int_{-\infty}^{\infty} |w'(t)|^2 \, dt}{\|w\|_2^4}$$

  - Using the Schwarz inequality we get

$$\Delta_w^2 \Delta_{\hat{w}}^2 \geq \frac{1}{\|w\|_2^4} \left[\int_{-\infty}^{\infty} |t\overline{w}(t) w'(t)| \, dt \right]^2$$
$$\geq \frac{1}{\|w\|_2^4} \left[\int_{-\infty}^{\infty} \frac{t}{2} [\overline{w}(t) w'(t) + \overline{w'}(t) w(t)] \, dt \right]^2$$
$$\geq \frac{1}{4\|w\|_2^4} \left[\int_{-\infty}^{\infty} t (|w(t)|^2)' \, dt \right]^2$$

Wavelets — Uncertainty principle (V)

- Proof of the Heisenberg Uncertainty Principle — continued
  - Integrating by parts and noting that $\lim_{|t| \to 0} \sqrt{t} f(t) = 0$ (since $|t|^{1/2} w(t) \in L_2(\mathbb{R})$ seen earlier) we obtain

$$\Delta_w^2 \Delta_{\hat{w}}^2 \geq \frac{1}{4\|w\|_2^4} \left[\int_{-\infty}^{\infty} |w(t)|^2 \, dt \right]^2 = \frac{1}{4}$$

  - An equality will be obtained when the Schwarz inequality becomes an equality; this implies that there exists $b \in \mathbb{C}$ such that

$$w'(t) = -2btw(t)$$

    so that there exists an $a \in \mathbb{C}$ such that $w(t) = ae^{-bt^2}$

- Thus the Gabor transform has the smallest possible time–frequency window.
- The above Heisenberg Uncertainty Principle has far–reaching consequences.

Integral Wavelet Transform (I)

- The short–time Fourier transform has a rigid time–frequency window, in the sense that its width (Δ_w) is unchanged for all frequencies analyzed; this turns out to be a limitation when studying functions with varying frequency content
- The Integral Wavelet Transform provides a window which:
  - automatically narrows when focusing on high frequencies,
  - automatically widens when focusing on low frequencies
- If $\psi \in L_2(\mathbb{R})$ satisfies the "admissibility" condition

$$C_\psi \triangleq \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \, d\omega < \infty,$$

  then ψ is called a basic wavelet. Relative to every basic wavelet ψ. the integral wavelet transform (IWT) in $L_2(\mathbb{R})$ is defined by

$$(W_\psi f)(a,b) \triangleq |a|^{\frac{1}{2}} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} \, dx, \quad f \in L_2(\mathbb{R}), \quad a \neq 0, b \in \mathbb{R},$$

Integral Wavelet Transform (II)

- Hereafter we will assume that $t\psi(t) \in L_2(\mathbb{R})$ and $\omega\hat{\psi}(\omega) \in L_2(\mathbb{R})$, so that the basic wavelet ψ provides a time-frequency window with finite area
- From the above assumption it also follows that $\hat{\psi}$ is a continuous function and therefore finiteness of C_ψ implies

$$\hat{\psi}(0) = 0 \implies \int_{-\infty}^{\infty} \psi(t)\,dt = 0$$

- Setting

$$\psi_{b;a}(t) \triangleq |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right),$$

the IWT can be written as $(W_\psi f)(b,a) = (f, \psi_{b;a})$

- If the wavelet ψ has the center and radius given by t^* and Δ_ψ, respectively, then the function $\psi_{b;a}$ has its center at $b + at^*$ and radius equal to $a\Delta_\psi$
- Thus, the IWT provides local information about the function f in a time window

$$[b + at^* - a\Delta_\psi, b + at^* + a\Delta_\psi]$$

which narrows down as $a \to 0$.

Integral Wavelet Transform (III)

- Consider the Fourier transform of a basic wavelet

$$\frac{1}{2\pi}\hat{\psi}_{b;a}(\omega) = \frac{|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\psi\left(\frac{t-b}{a}\right)dt = \frac{a|a|^{-\frac{1}{2}}}{2\pi}e^{-i\omega t}\hat{\psi}(\omega)$$

- Suppose that $\hat{\psi}$ has the center ω^* and radius $\Delta_{\hat{\psi}}$. Defining $\eta(\omega) \triangleq \hat{\psi}(\omega + \omega^*)$ we obtain w window function with center at the origin and unchanged radius
- Applying the Parseval identity to the definition of the IWT we obtain

$$(W_\psi f)(a,b) = \frac{a|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\overline{\eta(a\omega - \omega^*)}\,d\omega,$$

which, modulo multiplication by a constant factor and a linear frequency shift, localized information about the function f to the frequency window

$$\left[\frac{\omega^*}{a} - \frac{1}{a}\Delta_{\hat{\psi}}, \frac{\omega^*}{a} + \frac{1}{a}\Delta_{\hat{\psi}}\right]$$

Integral Wavelet Transform (IV)

- Note that the ratio of the center frequency ω^*/a to the bandwidth $2\Delta_{\hat{\psi}}/a$

$$\frac{\text{center frequency}}{\text{bandwidth}} = \frac{\omega^*}{2\Delta_{\hat{\psi}}}$$

is independent of the scaling a; thus, the bandwidth grows with frequency in an adaptive fashion (constant–Q filtering)

- Reconstruction of a function from its IWT
  Let ψ be a basic wavelet, then $\forall f, g \in L_2(\mathbb{R})$

$$\int_0^{\infty}\left[\int_{-\infty}^{\infty}(w_\psi f)(b,a)\overline{(w_\psi f)(b,a)}\,db\right]\frac{da}{a^2} = \frac{1}{2}C_\psi(f,g)$$

Furthermore, for any $f \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$ at which f is continuous

$$f(x) = \frac{2}{C_\psi}\int_0^{\infty}\left[\int_{-\infty}^{\infty}(w_\psi f)(b,a)\psi_{b;a}(x)\,db\right]\frac{da}{a^2}$$

Proof — using the Parseval identity, integrating with respect to da/a^2 and using the definition of C_ψ
Note the role of the admissibility condition for ψ

Discrete Wavelet Transform (I)

- Consider the IWT at a discrete set of samples $a = 2^{-j}$ and $b = k2^{-j}$ for some $j, k \in \mathbb{Z}$

$$(W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right) = \int_{-\infty}^{\infty} f(x)\overline{2^{j/2}\psi(2^j x - k)}\,dx = (f, \psi_{j,k})$$

where

$$\psi_{j,k} \triangleq 2^{j/2}\psi(2^j x - k)$$

must be chosen so that $\psi_{j,k}$ form a Riesz basis in $L_2(\mathbb{R})$ (i.e, the linear span of $\psi_{j,k}$ with $j, k \in \mathbb{Z}$ is dense in $L_2(\mathbb{R})$)

- If $\psi_{j,k}$ with $j, k \in \mathbb{Z}$ is a Riesz basis, the the relation

$$(\psi_{j,k}, \psi^{l,m}) = \delta_{j,l}\delta_{k,m}, \quad j,k,l,m \in \mathbb{Z}$$

uniquely defines another Riesz basis $\psi^{l,m}$ known as the dual basis

- Thus, every function $f \in L_2(\mathbb{R})$ has a unique representation

$$f(x) = \sum_{j,k=-\infty}^{\infty}(f, \psi_{j,k})\psi^{j,k}(x)$$

Discrete Wavelet Transform (II)

- For the above representation to qualify as a wavelet series, the the dual basis $\psi^{j,k}$ must be obtained from some basic wavelet $\tilde{\psi}$ by $\psi^{j,k}(x) = \tilde{\psi}_{j,k}(x)$, where

$$\tilde{\psi}_{j,k} \triangleq 2^{j/2}\tilde{\psi}(2^j x - k)$$

- In general, $\tilde{\psi}$ does not necessarily exist
- If ψ is chosen so that $\tilde{\psi}$ does exist, the pair $(\psi, \tilde{\psi})$ can be used interchangeably

$$f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k})\tilde{\psi}_{j,k}(x) = \sum_{j,k=-\infty}^{\infty} (f, \tilde{\psi}_{j,k})\psi_{j,k}(x)$$

- ψ and $\tilde{\psi}$ are called wavelet and dual wavelet, respectively
- If the basis $\psi_{j,k}$ is orthogonal, i.e., $\psi_{j,k} = \psi^{j,k}$ for $j,k \in \mathbb{Z}$, we obtain an orthogonal wavelet transform

$$f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k})\psi_{j,k}(x)$$

Discrete Wavelet Transform (III)

- Consider a wavelet ψ and the Riesz basis $\psi_{j,k}$ it generates; for each $j \in \mathbb{Z}$, let W_j denote the closure of the linear span of $\{\psi_{j,k} : k \in \mathbb{Z}\}$, i.e.,

$$W_j \triangleq \text{clos}_{L_2(\mathbb{R})}\{\psi_{j,k} : k \in \mathbb{Z}\}$$

- Evidently, $L_2(\mathbb{R})$ can be decomposed as a direct sum of the spaces W_j (dots over pluses indicate "direct sums")

$$L_2(\mathbb{R}) = \dot{\sum_{j\in\mathbb{Z}}} W_j \triangleq \cdots \dot{+} W_{-1} \dot{+} W_0 \dot{+} W_1 \dot{+} \ldots$$

and therefore every function $f \in L_2(\mathbb{R})$ has a unique decomposition

$$f(x) = \cdots + g_1(x) + g_0(x) + g_1(x) + \ldots$$

where $g_j \in W_j, \forall j \in \mathbb{Z}$

- if ψ is an orthogonal wavelet, then the subspaces $W_j \in L_2(\mathbb{R})$ are mutually orthogonal $W_j \perp W_l,\ j \neq l$ which means that

$$(g_j, g_l) = 0, \quad j \neq l$$

where $g_j \in W_j$ and $g_l \in W_l$

Discrete Wavelet Transform (IV)

- Therefore, in such case, the direct sum becomes an orthogonal sum

$$L_2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j \triangleq \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots$$

- Thus, an orthogonal wavelet ψ generates an orthogonal decomposition of the space $L_2(\mathbb{R})$, as the functions g_j are both unique and mutually orthogonal

Multiresolution Analysis (I)

- For every wavelet ψ (not necessarily orthogonal) we can consider the following space $V_j \in L_2(\mathbb{R}), \forall j \in \mathbb{Z}$

$$V_j = \cdots \dot{+} W_{j-2} \dot{+} W_{j-1}$$

- The subspaces V_j have the following very interesting properties:
  1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$
  2. $\text{clos}_{L_2}\left(\bigcup_{j\in\mathbb{Z}} V_j\right) = L_2(\mathbb{R})$
  3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$
  4. $V_{j+1} = V_j \dot{+} W_j,\ j \in \mathbb{Z}$
  5. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1},\ j \in \mathbb{Z}$
- Note that
  - In contrast to the subspaces W_j which satisfy $W_j \cap W_l = \{0\},\ j \neq l$, the sequence of subspaces V_j is nested $(1°)$
  - Every $f \in L_2(\mathbb{R})$ can be approximated arbitrarily accurately by its projections $P_j f$ on V_j $(2°)$

MULTIRESOLUTION ANALYSIS (II)

- If the reference subspace V_0 is generated by a single scaling function $\phi \in L_2(\mathbb{R})$ in the sense that

$$V_0 = \text{clos}_{L_2(\mathbb{R})}\{\phi_{0,k} : k \in \mathbb{Z}\}$$

where

$$\phi_{j,k} \triangleq 2^{j/2}\phi(2^j x - k),$$

then all the subspaces V_j are also generated by the same ϕ as

$$V_j = \text{clos}_{L_2(\mathbb{R})}\{\phi_{j,k} : k \in \mathbb{Z}\}$$

in the same way as the subspaces W_j are generated by the wavelet ψ

- In the multiresolution analysis at a given scale $(j+1)$
  - the subspace V_j represents the "large scale" features of the function
  - the subspaces W_j represents the "small scale" features (details) of the function

THE END