# PART V

# Wavelets & Multiresolution Analysis

# WAVELETS — OVERVIEW (I)

- What is wrong with Fourier analysis ???
  - All spatial information is hidden in the phases of the expansion coefficients and therefore not readily available
  - Localized functions ("bumps") tend to have a very complex representation in Fourier space
  - Local modification of the function affects its whole Fourier transform
  - If the dominant frequency changes in space, only average frequencies are encoded in Fourier coefficients
- Remedy — need an analysis tool that will encode both space (time) and frequency information at the same time
- Following the convention, will work with time ($t$) and frequency ($\omega$)

# WAVELETS — OVERVIEW (II)

- From Discrete Fourier Transform to Integral Fourier Transform — Consider the space $L_2(\mathbb{R})$ of square–integrable functions defined on $\mathbb{R}$; if $f \in L_2(\mathbb{R})$ satisfies suitable decay conditions at $\pm\infty$ (which??), the Discrete Fourier Transform can be replaced with the Integral Fourier Transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega t}\, dt$$

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega$$

- Interestingly, the Fourier Transforms (both discrete and integral) are constructed as "superpositions" of dilations of the function $w(x) = e^{ix}$ $(w_k(t) = w(kt))$
- Want to construct an integral transform using a basis function $\psi$ which is very localized (a "wavelet"); we will therefore need:
  - dilations
  - translations

# WAVELETS — GABOR TRANSFORM (I)

- The history begins with a windowed Fourier transform known as the Gabor Transform (1946)

$$(\mathcal{G}_b^{\alpha} f)(\omega) = \int_{-\infty}^{\infty} \left( f(t) e^{-i\omega t} \right) g_{\alpha}(t-b)\, dt,$$

where the window function is given by $g_{\alpha}(t) = \frac{1}{2\sqrt{\pi\alpha}} e^{-\frac{t^2}{4\alpha}}$ with $\alpha > 0$

- Note that the Fourier transform if a Gaussian function is another Gaussian function, i.e., $\int_{-\infty}^{\infty} e^{-i\omega x} e^{ax^2}\, dx = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$
- Note also that the window function has the following normalization $\int_{-\infty}^{\infty} g_{\alpha}(t-b)\, db = \int_{-\infty}^{\infty} g_{\alpha}(x)\, dx = 1$
- Therefore, for the Gabor transform we obtain

$$\int_{-\infty}^{\infty} (\mathcal{G}_b^{\alpha} f)(\omega)\, db = \hat{f}(\omega), \quad \omega \in \mathbb{R}$$

- Thus, the set $\{\mathcal{G}_b^{\alpha} f : b \in \mathbb{R}\}$ of Gabor transforms of $f$ decomposes the Fourier transforms $\hat{f}$ of $f$ exactly to give its local spectral information

# WAVELETS — GABOR TRANSFORM (II)

- The width of the window function can be characterized by employing the notion of the standard deviation

$$\Delta_{g_\alpha} \triangleq \frac{1}{\|g_\alpha\|_2} \left\{ \int_{-\infty}^{\infty} x^2 g_\alpha^2(x)\, dx \right\}^{1/2}$$

- Note that for $\alpha > 0$ $\Delta_{g_\alpha} = \sqrt{\alpha}$
  Proof:
  - $\|g_\alpha\| = (8\pi\alpha)^{-1/4}$ can be evaluated setting $\omega = 0$ and $a = (2\alpha)^{-1}$ in the expression for the Fourier transform of a Gaussian function
  - $\int_{-\infty}^{\infty} x^2 g_\alpha^2(x)\, dx$ can be evaluated twice differentiating the Fourier transform of a Gaussian function and again setting $\omega = 0$ and $a = (2\alpha)^{-1}$

- Instead of localizing the Fourier transform of $f$, the Gabor transform may equivalently be regarded as windowing $f$ with the window function $\mathcal{G}_{b,\omega}^{\alpha}$

$$(\mathcal{G}_b^\alpha f)(\omega) = (f, \mathcal{G}_{b,\omega}^{\alpha}) = \int_{-\infty}^{\infty} f(t)\overline{\mathcal{G}_{b,\omega}^{\alpha}(t)}\, dt$$

# WAVELETS — GABOR TRANSFORM (III)

- Using the Parseval identity and noting that

$$\hat{\mathcal{G}}^{\alpha}_{b,\omega}(\eta) = e^{-ib(\eta-\omega)} e^{-\alpha(\eta-\omega)^2}$$

we obtain for the Gabor transform

$$\begin{aligned}
(\mathcal{G}^{\alpha}_{b} f)(\omega) &= (f, \mathcal{G}^{\alpha}_{b,\omega}) = \frac{1}{2\pi}(\hat{f}, \hat{\mathcal{G}}^{\alpha}_{b,\omega}) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\eta) e^{ib(\eta-\omega)} e^{-\alpha(\eta-\omega)^2}\, d\eta \\
&= \frac{e^{-ib\omega}}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} \left(e^{ib\eta} \hat{f}(\eta)\right) g_{1/4\alpha}(\eta-\omega)\, d\eta \\
&= \frac{e^{-ib\omega}}{2\sqrt{\pi\alpha}} (\mathcal{G}^{1/4\alpha}_{\omega} \hat{f})(-b)
\end{aligned}$$

- The third line ( in red ) indicates that up to a multiplicative factor $\sqrt{\frac{\pi}{\alpha}} e^{-ib\omega}$
  - the windowed Fourier transform of $f$ with $g_\alpha$ at $t = b$,
  - the window inverse Fourier transform of $\hat{f}$ with $g_{1/4\alpha}$ at $\eta = \omega$

  are equal!

# WAVELETS — UNCERTAINTY PRINCIPLE (I)

- Consider more general window functions $w \in L_2(\mathbb{R})$ which satisfy the requirement

$$tw(t) \in L_2(\mathbb{R})$$

It can be shown that

  – $|t|^{1/2} w(t) \in L_2(\mathbb{R})$
  – $w \in L_1(\mathbb{R})$
  – the Fourier transform $\hat{w}$ is continuous
  – $\hat{w} \in L_2(\mathbb{R})$

Note, however, that in general $x\hat{w}(x) \notin L_2(\mathbb{R})$, therefore $w$ may not in general be a frequency window function

- If $w \in L_2(\mathbb{R})$ is chosen so that both $w$ and $\hat{w}$ satisfy the above condition, then the window Fourier transform

$$(\tilde{\mathcal{G}}_b f)(\omega) = \int_{-\infty}^{\infty} \left( f(t) e^{-i\omega t} \right) \overline{w(t-b)}\, dt = (f, W_{b,\omega}),$$

where $W_{b,\omega} = e^{i\omega t} w(t-b)$, is called a short–time Fourier transform

# WAVELETS — UNCERTAINTY PRINCIPLE (II)

- We can define the center $x^*$ and radius $\Delta_w$ of $w$ as

$$x^* \triangleq \frac{1}{\|w\|_2^2} \int_{-\infty}^{\infty} t|w(t)|^2\, dt, \qquad \Delta_w \triangleq \frac{1}{\|w\|_2} \left\{ \int_{-\infty}^{\infty} (t-x^*)^2 |w(t)|^2\, dt \right\}^{1/2}$$

- Then, $(\tilde{\mathcal{G}}_b f)(\omega)$ gives local information on $f$ in the time–window

$$[x^* + b - \Delta_w, x^* + b + \Delta_w]$$

- We can determine the center $\omega^*$ and the radius $\Delta_{\hat{w}}$ of the (frequency) window function $\hat{w}$ using formulae similar to the above

- Defining $V_{b,\omega}(\eta) \triangleq \frac{1}{2\pi}\hat{W}_{b,\omega}(\eta) = \frac{1}{2\pi} e^{ib\omega} e^{-ib\eta} \hat{w}(\eta - \omega)$, which is also a window function with the center $\omega^* + \omega$ and radius $\Delta_{\hat{w}}$ we can write (using the Parseval identity)

$$(\tilde{\mathcal{G}}_b f)(\omega) = (f, W_{b,\omega}) = (\hat{f}, V_{b,\omega})$$

- Thus, $(\tilde{\mathcal{G}}_b f)(\omega)$ also gives local spectral information about $t$ in the frequency window

$$[\omega^* + \omega - \Delta_{\hat{w}}, \omega^* + \omega + \Delta_{\hat{w}}]$$

# WAVELETS — UNCERTAINTY PRINCIPLE (III)

- Therefore by choosing $w \in L_2(\mathbb{R})$ such that both $xw(x) \in L_2(\mathbb{R})$ and $x\hat{w}(x) \in L_2(\mathbb{R})$ to define a windowed Fourier transform $(\tilde{\mathcal{G}}_b f)(\omega)$ we obtain localization in a time–frequency window

$$[x^* + b - \Delta_w, x^* + b + \Delta_w] \times [\omega^* + \omega - \Delta_{\hat{w}}, \omega^* + \omega + \Delta_{\hat{w}}]$$

  with area equal to $4\Delta_w\Delta_{\hat{w}}$

- In fact, there is a relation between possible time and frequency windows which is made precise in the following theorem

- Heisenberg Uncertainty Principle — Let $w \in L_2(\mathbb{R})$ be chosen so that $xw(x) \in L_2(\mathbb{R})$ and $x\hat{w}(x) \in L_2(\mathbb{R})$. Then

$$\Delta_w\Delta_{\hat{w}} \geq \frac{1}{2}$$

  Furthermore, equality is attained if and only iff

$$w(t) = ce^{i\alpha t} g_\alpha(t - b),$$

  where $c \neq 0$, $\alpha > 0$, and $a, b \in \mathbb{R}$.

# WAVELETS — UNCERTAINTY PRINCIPLE (IV)

- Proof of the Heisenberg Uncertainty Principle
  - Let us assume that the centers $x^*$ and $\omega^*$ are zero (if they are not, then we can modify $w$ as $\tilde{w}(t) = e^{-i\omega^* t} f(t + x^*)$)
  - We observe that

$$\Delta_w^2 \Delta_{\hat{w}}^2 = \frac{\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt \int_{-\infty}^{\infty} \omega^2 |\hat{w}(\omega)|^2 \, d\omega}{\|w\|_2^2 \|\hat{w}\|_2^2}$$

$$= \frac{\int_{-\infty}^{\infty} t^2 |w(t)|^2 \, dt \int_{-\infty}^{\infty} |w'(t)|^2 \, dt}{\|w\|_2^4}$$

  - Using the Schwarz inequality we get

$$\Delta_w^2 \Delta_{\hat{w}}^2 \geq \frac{1}{\|w\|_2^4} \left[ \int_{-\infty}^{\infty} |t\overline{w}(t) w'(t)| \, dt \right]^2$$

$$\geq \frac{1}{\|w\|_2^4} \left[ \int_{-\infty}^{\infty} \frac{t}{2} [\overline{w}(t) w'(t) + \overline{w'}(t) w(t)] \, dt \right]^2$$

$$\geq \frac{1}{4\|w\|_2^4} \left[ \int_{-\infty}^{\infty} t (|w(t)|^2)' \, dt \right]^2$$

# WAVELETS — UNCERTAINTY PRINCIPLE (V)

- Proof of the Heisenberg Uncertainty Principle — continued
  - Integrating by parts and noting that $\lim_{|t|\to 0} \sqrt{t}\, f(t) = 0$ (since $|t|^{1/2} w(t) \in L_2(\mathbb{R})$ seen earlier) we obtain

$$\Delta_w^2 \Delta_{\hat{w}}^2 \geq \frac{1}{4\|w\|_2^4} \left[ \int_{-\infty}^{\infty} |w(t)|^2 \, dt \right]^2 = \frac{1}{4}$$

  - An equality will be obtained when the Schwarz inequality becomes an equality; this implies that there exists $b \in \mathbb{C}$ such that

$$w'(t) = -2btw(t)$$

    so that there exists an $a \in \mathbb{C}$ such that $w(t) = ae^{-bt^2}$

- Thus the Gabor transform has the smallest possible time–frequency window.
- The above Heisenberg Uncertainty Principle has far–reaching consequences.

# INTEGRAL WAVELET TRANSFORM (I)

- The short–time Fourier transform has a rigid time–frequency window, in the sense that its width ($\Delta_w$) is unchanged for all frequencies analyzed; this turns out to be a limitation when studying functions with varying frequency content
- The Integral Wavelet Transform provides a window which:
  - automatically narrows when focusing on high frequencies,
  - automatically widens when focusing on low frequencies
- If $\psi \in L_2(\mathbb{R})$ satisfies the "admissibility" condition

$$C_\psi \triangleq \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty,$$

then $\psi$ is called a basic wavelet . Relative to every basic wavelet $\psi$. the integral wavelet transform (IWT) in $L_2(\mathbb{R})$ is defined by

$$(W_\psi f)(a,b) \triangleq |a|^{\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\psi\left(\frac{x-b}{a}\right)} dx, \quad f \in L_2(\mathbb{R}), \quad a \neq 0, b \in \mathbb{R},$$

# INTEGRAL WAVELET TRANSFORM (II)

- Hereafter we will assume that $t\psi(t) \in L_2(\mathbb{R})$ and $\omega\hat{\psi}(\omega) \in L_2(\mathbb{R})$, so that the basic wavelet $\psi$ provides a time-frequency window with finite area
- From the above assumption it also follows that $\hat{\psi}$ is a continuous function and therefore finiteness of $C_\psi$ implies

$$\hat{\psi}(0) = 0 \implies \int_{-\infty}^{\infty} \psi(t)\, dt = 0$$

- Setting

$$\psi_{b;a}(t) \triangleq |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right),$$

  the IWT can be written as $(W_\psi f)(b,a) = (f, \psi_{b;a})$
- If the wavelet $\psi$ has the center and radius given by $t^*$ and $\Delta_\psi$, respectively, then the function $\psi_{b;a}$ has its center at $b + at^*$ and radius equal to $a\Delta_\psi$
- Thus, the IWT provides local information about the function $f$ in a time window

$$[b + at^* - a\Delta_\psi, b + at^* + a\Delta_\psi]$$

  which narrows down as $a \to 0$.

# INTEGRAL WAVELET TRANSFORM (III)

- Consider the Fourier transform of a basic wavelet

$$\frac{1}{2\pi}\hat{\psi}_{b;a}(\omega) = \frac{|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\psi\left(\frac{t-b}{a}\right)dt = \frac{a|a|^{-\frac{1}{2}}}{2\pi}e^{-i\omega t}\hat{\psi}(\omega)$$

- Suppose that $\hat{\psi}$ has the center $\omega^*$ and radius $\Delta_{\hat{\psi}}$. Defining $\eta(\omega) \triangleq \hat{\psi}(\omega+\omega^*)$ we obtain w window function with center at the origin and unchanged radius

- Applying the Parseval identity to the definition of the IWT we obtain

$$(W_\psi f)(a,b) = \frac{a|a|^{-\frac{1}{2}}}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega t}\overline{\eta(a\omega-\omega^*)}\,d\omega,$$

  which, modulo multiplication by a constant factor and a linear frequency shift, localized information about the function $f$ to the frequency window

$$\left[\frac{\omega^*}{a}-\frac{1}{a}\Delta_{\hat{\psi}},\frac{\omega^*}{a}+\frac{1}{a}\Delta_{\hat{\psi}}\right]$$

# Integral Wavelet Transform (IV)

- Note that the ratio of the center frequency $\omega^*/a$ to the bandwidth $2\Delta_{\hat{\psi}}/a$

$$\frac{\text{center frequency}}{\text{bandwidth}} = \frac{\omega^*}{2\Delta_{\hat{\psi}}}$$

  is independent of the scaling $a$; thus, the bandwidth grows with frequency in an adaptive fashion ( constant–Q filtering )

- Reconstruction of a function from its IWT
  Let $\psi$ be a basic wavelet, then $\forall f, g \in L_2(\mathbb{R})$

$$\int_0^\infty \left[ \int_{-\infty}^\infty (w_\psi f)(b,a) \overline{(w_\psi f)(b,a)}\, db \right] \frac{da}{a^2} = \frac{1}{2} C_\psi(f,g)$$

  Furthermore, for any $f \in L_2(\mathbb{R})$ and $x \in \mathbb{R}$ at which $f$ is continuous

$$f(x) = \frac{2}{C_\psi} \int_0^\infty \left[ \int_{-\infty}^\infty (w_\psi f)(b,a) \psi_{b;a}(x)\, db \right] \frac{da}{a^2}$$

  Proof — using the Parseval identity, integrating with respect to $da/a^2$ and using the definition of $C_\psi$
  Note the role of the admissibility condition for $\psi$

# DISCRETE WAVELET TRANSFORM (I)

- Consider the IWT at a discrete set of samples $a = 2^{-j}$ and $b = k2^{-j}$ for some $j, k \in \mathbb{Z}$

$$(W_\psi f)\left(\frac{k}{2^j}, \frac{1}{2^j}\right) = \int_{-\infty}^{\infty} f(x)\overline{2^{j/2}\psi(2^j x - k)}\, dx = (f, \psi_{j,k})$$

  where

$$\psi_{j,k} \triangleq 2^{j/2}\psi(2^j x - k)$$

  must be chosen so that $\psi_{j,k}$ form a Riesz basis in $L_2(\mathbb{R})$ (i.e, the linear span of $\psi_{j,k}$ with $j, k \in \mathbb{Z}$ is dense in $L_2(\mathbb{R})$)

- If $\psi_{j,k}$ with $j, k \in \mathbb{Z}$ is a Riesz basis, the the relation

$$(\psi_{j,k}, \psi^{l,m}) = \delta_{j,l}\delta_{k,m}, \quad j, k, l, m \in \mathbb{Z}$$

  uniquely defines another Riesz basis $\psi^{l,m}$ known as the dual basis

- Thus, every function $f \in L_2(\mathbb{R})$ has a unique representation

$$f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k})\psi^{j,k}(x)$$

# DISCRETE WAVELET TRANSFORM (II)

- For the above representation to qualify as a wavelet series , the the dual basis $\psi^{j,k}$ must be obtained from some basic wavelet $\tilde{\psi}$ by $\psi^{j,k}(x) = \tilde{\psi}_{j,k}(x)$ , where

$$\tilde{\psi}_{j,k} \triangleq 2^{j/2}\tilde{\psi}(2^j x - k)$$

- In general, $\tilde{\psi}$ does not necessarily exist
- If $\psi$ is chosen so that $\tilde{\psi}$ does exist, the pair $(\psi, \tilde{\psi})$ can be used interchangeably

$$f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k})\tilde{\psi}_{j,k}(x) = \sum_{j,k=-\infty}^{\infty} (f, \tilde{\psi}_{j,k})\psi_{j,k}(x)$$

- $\psi$ and $\tilde{\psi}$ are called wavelet and dual wavelet , respectively
- If the basis $\psi_{j,k}$ is orthogonal, i.e., $\psi_{j,k} = \psi^{j,k}$ for $j,k \in \mathbb{Z}$, we obtain an orthogonal wavelet transform

$$f(x) = \sum_{j,k=-\infty}^{\infty} (f, \psi_{j,k})\psi_{j,k}(x)$$

# DISCRETE WAVELET TRANSFORM (III)

- Consider a wavelet $\psi$ and the Riesz basis $\psi_{j,k}$ it generates; for each $j \in \mathbb{Z}$, let $W_j$ denote the closure of the linear span of $\{\psi_{j,k} : k \in \mathbb{Z}\}$, i.e.,

$$W_j \triangleq \mathrm{clos}_{L_2(\mathbb{R})}\{\psi_{j,k} : k \in \mathbb{Z}\}$$

- Evidently, $L_2(\mathbb{R})$ can be decomposed as a direct sum of the spaces $W_j$ (dots over pluses indicate "direct sums")

$$L_2(\mathbb{R}) = \dot{\sum_{j \in \mathbb{Z}}} W_j \triangleq \dots \dot{+} W_{-1} \dot{+} W_0 \dot{+} W_1 \dot{+} \dots$$

  and therefore every function $f \in L_2(\mathbb{R})$ has a unique decomposition

$$f(x) = \dots + g_1(x) + g_0(x) + g_1(x) + \dots$$

  where $g_j \in W_j, \forall j \in \mathbb{Z}$

- if $\psi$ is an orthogonal wavelet, then the subspaces $W_j \in L_2(\mathbb{R})$ are mutually orthogonal $W_j \perp W_l,\ j \neq l$ which means that

$$(g_j, g_l) = 0, \quad j \neq l$$

  where $g_j \in W_j$ and $g_l \in W_l$

# DISCRETE WAVELET TRANSFORM (IV)

- Therefore, in such case, the direct sum becomes an orthogonal sum

$$L_2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j \triangleq \cdots \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus \ldots$$

- Thus, an orthogonal wavelet $\psi$ generates an orthogonal decomposition of the space $L_2(\mathbb{R})$, as the functions $g_j$ are both unique and mutually orthogonal

# MULTIRESOLUTION ANALYSIS (I)

- For every wavelet $\psi$ (not necessarily orthogonal) we can consider the following space $V_j \in L_2(\mathbb{R}), \forall j \in \mathbb{Z}$

$$V_j = \cdots \dot{+} W_{j-2} \dot{+} W_{j-1}$$

- The subspaces $V_j$ have the following very interesting properties:
  1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$
  2. $\mathrm{clos}_{L_2}\left(\bigcup_{j\in\mathbb{Z}} V_j\right) = L_2(\mathbb{R})$
  3. $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$
  4. $V_{j+1} = V_j \dot{+} W_j,\ j \in \mathbb{Z}$
  5. $f(x) \in V_j \ \Leftrightarrow f(2x) \in V_{j+1},\ j \in \mathbb{Z}$
- Note that
  - In contrast to the subspaces $W_j$ which satisfy $W_j \cap W_l = \{0\},\ j \neq l$, the sequence of subspaces $V_j$ is nested ($1^\circ$)
  - Every $f \in L_2(\mathbb{R})$ can be approximated arbitrarily accurately by its projections $P_j f$ on $V_j$ ($2^\circ$)

# MULTIRESOLUTION ANALYSIS (II)

- If the reference subspace $V_0$ is generated by a single scaling function $\phi \in L_2(\mathbb{R})$ in the sense that

$$V_0 = \text{clos}_{L_2(\mathbb{R})}\{\phi_{0,k} : k \in \mathbb{Z}\}$$

where

$$\phi_{j,k} \triangleq 2^{j/2}\phi(2^j x - k),$$

then all the subspaces $V_j$ are also generated by the same $\phi$ as

$$V_j = \text{clos}_{L_2(\mathbb{R})}\{\phi_{j,k} : k \in \mathbb{Z}\}$$

in the same way as the subspaces $W_j$ are generated by the wavelet $\psi$

- In the multiresolution analysis at a given scale $(j+1)$
  - the subspace $V_j$ represents the "large scale" features of the function
  - the subspaces $W_j$ represents the "small scale" features (details) of the function

THE END