Universal Algebra and CSP: The Basics

Andrei Krokhin

Durham University

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Forms of CSP: reminder

- Variable-value

  Given finite sets V (variables), D (values), and a set of constraints $\{(\overline{s}_1, R_1), \ldots, (\overline{s}_q, R_q)\}$ over V, is there a function $\varphi : V \to D$ such that $\varphi(\overline{s}_i) \in R_i$ for all i?

- Satisfiability

  Given a formula $\mathcal{P}(x_1, \ldots, x_n) = R_1(\overline{s}_1) \wedge \ldots \wedge R_q(\overline{s}_q)$ (where R_i's are seen as predicates), is $\mathcal{P}$ satisfiable?

- Homomorphism

  Given two finite similar relational structures, $\mathcal{A} = (V;\ R_1^{\mathcal{A}}, \ldots, R_k^{\mathcal{A}})$ and $\mathcal{B} = (D;\ R_1^{\mathcal{B}}, \ldots, R_k^{\mathcal{B}})$, is there a homomorphism $h : \mathcal{A} \to \mathcal{B}$?

Outline

- CSP classification problem
- From CSPs to clones
- From clones to algebras
- From algebras to varieties
- Coffee break
- Hardness results
- Tractability results
- Partial classification results

Constraint Languages

Fix a finite set D.

Definition 1 *A constraint language is any set* Γ *of relations on* D*. The problem* CSP(Γ) *is the restriction of* CSP *where all constraint relations* R_i *must belong to* Γ*.*

Equivalent problem: for finite Γ, CSP(Γ) is the same as CSP($\mathcal{B}$) $= \{\mathcal{A} \mid \mathcal{A} \to \mathcal{B}\}$, where the structure $\mathcal{B}$ is obtained from Γ by indexing relations.

Examples

Many standard computational problems can be expressed as CSP(Γ) for suitable Γ:

- Let $D = \{0,1\}$ and $R = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\}$. If $\Gamma = \{R\}$ then CSP(Γ) is NOT-ALL-EQUAL SAT. This problem is **NP**-complete.
- If $\Gamma = \{\neq_D\}$ where $\neq_D$ is the disequality relation on D and $|D| = k$ then CSP(Γ) is GRAPH k-COLOURING. Tractable if $k \leq 2$, **NP**-complete for $k \geq 3$.
- For a semigroup S on D, let $R_S = \{(x,y,z) \mid xy = z\}$. If $\Gamma = \{R_S\} \cup \{\{d\} \mid d \in D\}$ then CSP(Γ) is the problem of solving SYSTEMS OF EQUATIONS over S.

Classification problems

Two main classification problems about problems CSP(Γ):

1. Classify CSP(Γ) w.r.t. computational complexity, (i.e., w.r.t. membership in a given complexity class)
2. Classify CSP($\mathcal{B}$) w.r.t. descriptive complexity, (i.e., w.r.t. definability in a given logic)

Conjecture 1 (Feder, Vardi, 1998)
Dichotomy Conjecture: for each (finite) Γ, *the problem* CSP(Γ) *is either tractable (i.e., in* $\mathbf{P}$*) or* $\mathbf{NP}$*-complete.*

Relational Clones

Definition 2 *For a set of relations Γ on D, let $\langle\Gamma\rangle$ denote the set of all relations that can be expressed by primitive positive (p.p.-) formulas over Γ, that is, using*

- *relations in $\Gamma \cup \{=_D\}$,*
- *conjunction,*
- *existential quantification.*

Example: $R_1(x, y, z) = \exists u[R_2(x, u) \wedge R_3(u, y) \wedge y = z]$.

The set $\langle\Gamma\rangle$ is the relational clone generated by Γ.

Relational Clones cont'd

Theorem 1 (Jeavons, 1998) *If Γ_1 and Γ_2 are finite constraint languages and $\langle\Gamma_1\rangle \subseteq \langle\Gamma_2\rangle$ then* CSP(Γ_1) *is polynomial-time reducible to* CSP(Γ_2).

Proof. Reduction goes as follows:

1. Take an instance $R_1(\overline{s}_1) \wedge \ldots \wedge R_q(\overline{s}_q)$ where $R_i \in \Gamma_1$.
2. Since $R_i \in \langle\Gamma_2\rangle$, replace each $R_i(\overline{s}_1)$ by the corresponding p.p.-formula over Γ_2
3. Remove quantifiers, renaming variables as necessary.
4. Identify variables connected by equality constraints.
5. Remove equality constraints.

Example

Assume $\Gamma_1 = \{R_1\}$ and $\Gamma_2 = \{R_2, R_3\}$, and $R_1 \in \langle \Gamma_2 \rangle$.
0) Fix expression for R_1, for example,
$R_1(x,y,z) = \exists u[R_2(x,u) \wedge R_3(u,y) \wedge y = z]$.

1) Take an instance $R_1(x,y,z) \wedge R_1(z,t,y)$.

2) Transform it into equivalent formula
$\exists u[R_2(x,u) \wedge R_3(u,y) \wedge y = z] \wedge \exists u[R_2(z,u) \wedge R_3(u,t) \wedge t = y]$.

3) Remove quantifiers, renaming the quantified variables
$R_2(x,u_1) \wedge R_3(u_1,y) \wedge y = z \wedge R_2(z,u_2) \wedge R_3(u_2,t) \wedge t = y$.

4-5) Identify z, t with y and remove equality constraints
$R_2(x,u_1) \wedge R_3(u_1,y) \wedge R_2(y,u_2) \wedge R_3(u_2,y)$.

Invariance and Polymorphisms

Definition 3 *An m-ary relation R is invariant under an n-ary operation f (or f is a polymorphism of R) if, for any tuples $\bar{a}_1 = (a_{11}, \ldots, a_{1m}), \ldots, \bar{a}_n = (a_{n1}, \ldots, a_{nm}) \in R$, the tuple obtained by applying f componentwise belongs to R.*

$$
\begin{array}{cccccccc}
 & f & & f & & f & & \\
(& a_{11} & , & \ldots & , & a_{1m} &) & \in R \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
(& a_{n1} & , & \ldots & , & a_{nm} &) & \in R \\
\hline
(& f(a_{11}, \ldots, a_{n1}) & , & \ldots & , & f(a_{1m}, \ldots, a_{nm}) &) & \in R
\end{array}
$$

Galois Correspondence

Let $\mathrm{Pol}(\Gamma)$ be the set of all polymorphisms of Γ.

If F is a set of operations on D, let
$\mathrm{Inv}(F) = \{R \mid R$ is invariant under all operations in $F\}$,
and let $\langle F \rangle$ be the set of all operations obtained from F via superpositions $f(f_1, \ldots, f_n)$.
Then $\langle F \rangle$ is called the clone generated by F.

Theorem 2 (Geiger, 1968; Bodnarchuk et al., 1969)

- *For any constraint language* Γ, $\langle \Gamma \rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma))$.
- *For any set* F *of operations,* $\langle F \rangle = \mathrm{Pol}(\mathrm{Inv}(F))$.

Polymorphisms and Complexity

Theorem 3 (Jeavons, 1998) *Let* Γ_1 *and* Γ_2 *be finite constraint languages. If* $\mathrm{Pol}(\Gamma_1) \subseteq \mathrm{Pol}(\Gamma_2)$ *then* $\mathrm{CSP}(\Gamma_2)$ *is polynomial-time reducible to* $\mathrm{CSP}(\Gamma_1)$*.*

Proof. The operator Inv() is anti-monotone, so $\mathrm{Pol}(\Gamma_1) \subseteq \mathrm{Pol}(\Gamma_2)$ implies

$$\langle \Gamma_2 \rangle = \mathrm{Inv}(\mathrm{Pol}(\Gamma_2)) \subseteq \mathrm{Inv}(\mathrm{Pol}(\Gamma_1)) = \langle \Gamma_1 \rangle.$$

We already know that $\langle \Gamma_2 \rangle \subseteq \langle \Gamma_1 \rangle$ implies the conclusion of the theorem.

Corollary 1 *If* $\mathrm{Pol}(\Gamma_1) = \mathrm{Pol}(\Gamma_2)$ *then* $\mathrm{CSP}(\Gamma_1)$ *and* $\mathrm{CSP}(\Gamma_2)$ *are polynomial time equivalent.*

Technical Definition

Definition 4 *We shall say that* Γ *is tractable if, for any finite* $\Gamma' \subseteq \Gamma$, $\mathrm{CSP}(\Gamma')$ *is tractable.*
If $\mathrm{CSP}(\Gamma)$ *is tractable then we call* Γ *globally tractable.*
We say that Γ *is* **NP***-complete if* $\mathrm{CSP}(\Gamma')$ *is* **NP***-complete for some finite* $\Gamma' \subseteq \Gamma$*.*

Fact. All known tractable constraint languages are globally tractable.

It has been conjectured that this is always the case.

Reducing the Domain

For a unary operation f and a relation R on D, let
$f(R) = \{(f(a_1), \dots, f(a_n)) \mid (a_1, \dots, a_n) \in R\}$.
For a constraint language Γ, let $f(\Gamma) = \{f(R) \mid R \in \Gamma\}$.

Theorem 4 (Jeavons, 1998) *Let Γ be finite, and let $f \in \mathrm{Pol}(\Gamma)$ be unary with minimal range. Then $\mathrm{CSP}(\Gamma)$ and $\mathrm{CSP}(f(\Gamma))$ are polynomial-time equivalent.*

Proof. Take an instance $\mathcal{P} = \bigwedge R_i(\overline{s}_i)$ of $\mathrm{CSP}(\Gamma)$ and consider the instance $\mathcal{P}' = \bigwedge f(R_i)(\overline{s}_i)$ of $\mathrm{CSP}(f(\Gamma))$.

Since $f(R_i) \subseteq R_i$, we have $Sol(\mathcal{P}') \subseteq Sol(\mathcal{P})$, and conversely, for each $\varphi \in Sol(\mathcal{P})$, $f \circ \varphi$ is a solution to $\mathcal{P}'$.

Mapping $\mathcal{P}' \mapsto \mathcal{P}$ is the reduction in the other direction.

Adding the Constants

By previous slide, assume that unary operations in $\mathrm{Pol}(\Gamma)$ form a permutation group G.

Theorem 5 (Bulatov, Jeavons, K, 2005)
Let $\Gamma' = \Gamma \cup \{\{a\} \mid a \in D\}$. *Then* $\mathrm{CSP}(\Gamma)$ *and* $\mathrm{CSP}(\Gamma')$ *are polynomial-time equivalent.*

Proof. Obviously, $\mathrm{CSP}(\Gamma)$ reduces to $\mathrm{CSP}(\Gamma')$.

The other direction. Let $D = \{a_1, \ldots, a_n\}$. Then $R_G \in \langle\Gamma\rangle$ where

$$R_G = \{(g(a_1), \ldots, g(a_n)) \mid g \in G\}.$$

We may assume that $R_G \in \Gamma$ and $=_D \in \Gamma$.

Proof cont'd

Take an instance $\mathcal{P}'$ of CSP(Γ') over a set of variables V' and build an equivalent instance $\mathcal{P}$ of CSP(Γ) as follows.

- Include all constraints from $\mathcal{P}'$ to $\mathcal{P}$
- Introduce new variables $y_a, a \in D$
- Replace each constraint of the form $x = a$ with $x = y_a$
- Introduce new constraint $R_G(y_{a_1}, \ldots, y_{a_n})$

Any solution of $\mathcal{P}'$ extends to a solution of $\mathcal{P}$ by $y_{a_i} \mapsto a_i$.

If ϕ is a solution to $\mathcal{P}$ then we have
$\phi(y_{a_1}, \ldots, y_{a_n}) = (g(a_1), \ldots, g(a_n))$ for some $g \in G$.
Then $g^{-1} \circ \phi$ (restricted to V') is a solution to $\mathcal{P}'$.

Search Problem

Theorem 6 (Bulatov, Jeavons, K, 2005)
If the decision problem CSP(Γ) *is tractable then the corresponding search problem is tractable as well.*

Proof. Take an instance $\mathcal{P}$ of CSP(Γ) and build an equivalent instance $\mathcal{P}'$ of CSP($f(\Gamma)$) s.t. $Sol(\mathcal{P}') \subseteq Sol(\mathcal{P})$.

Remember: CSP($f(\Gamma) \cup \{\{a\} \mid a \in f(D)\}$) is tractable.

For all variables x (in order)

for all values $a \in f(D)$

if $\mathcal{P}' \wedge (x = a)$ is satisfiable

set $\mathcal{P}' := \mathcal{P}' \wedge (x = a)$ and go to next variable

From CSP to Algebras

Definition 5 *A finite algebra is a pair* $\mathbf{A} = (D, F)$ *where* D *is a finite set and* F *is a family of operations on* D.

The clone $\langle F \rangle$ *is called the clone of term operations of* $\mathbf{A}$. *Two algebras* $\mathbf{A}_1 = (D, F_1)$ *and* $\mathbf{A}_2 = (D, F_2)$ *are said to be term equivalent if they have the same clone of term operations.*

Definition 6 *Let* $\mathbf{A} = (A, F)$ *be a finite algebra.*
Let $\mathrm{CSP}(\mathbf{A})$ *denote* $\mathrm{CSP}(\mathrm{Inv}(F))$.
We say that $\mathbf{A}$ *is tractable, or globally tractable, or* **NP***-complete if* $\mathrm{CSP}(\mathbf{A})$ *has the corresponding property.*

Note: Term equivalent algebras have the same complexity.

Varieties

Definition 7 *For a class $\mathcal{K}$ of similar algebras, let*

- $\mathsf{H}(\mathcal{K})$ *be the class of all hom images of algebras from* $\mathcal{K}$
- $\mathsf{S}(\mathcal{K})$ *be the class of all subalgebras of algebras from* $\mathcal{K}$
- $\mathsf{P}(\mathcal{K})$ *and* $\mathsf{P}_{fin}(\mathcal{K})$ *be the classes of all and all finite, respectively, direct products of algebras from* $\mathcal{K}$

A class of similar algebras that is closed under the operators H, S *and* P *is called a variety.*

For an algebra $\mathbf{A}$, *the class* $\mathsf{HSP}(\mathbf{A})$ *is the variety generated by* $\mathbf{A}$, *and is denoted* $\mathrm{var}(\mathbf{A})$.

A View on CSP(A)

Fact. Relations from $\mathrm{Inv}(F_A)$ are universes of algebras from $\mathsf{SP}_{fin}(\mathbf{A})$ (the so-called subpowers of $\mathbf{A}$).

Take an instance $\{(\overline{s}_1, R_1), \ldots, (\overline{s}_q, R_q)\}$ of $\mathrm{CSP}(\mathbf{A})$, over a set of variables $V = \{x_1, \ldots, x_n\}$.

For a constraint $(\overline{s}_i, R_i)$, consider the following subalgebra $\mathbf{A}_i$ of $\mathbf{A}^V$: $\{\overline{a} \in A^V \mid \mathrm{pr}_{\overline{s}_i} \overline{a} \in R_i\}$.

Solutions to the instance = elements in $\bigcap_{i=1}^{q} \mathbf{A}_i$.

Hence, $\mathrm{CSP}(\mathbf{A})$ = Subalgebra Intersection problem: "given" subalgebras $\mathbf{A}_1, \ldots, \mathbf{A}_q$ of $\mathbf{A}^k$, $k \geq 1$, is it true that $\bigcap_{i=1}^{q} \mathbf{A}_i \neq \emptyset$?

From Algebras to Varieties

Theorem 7 (Bulatov, Jeavons, 2003) *If an algebra* $\mathbf{A}$ *is tractable then every finite algebra in* $\mathrm{var}(\mathbf{A})$ *is tractable. If* $\mathrm{var}(\mathbf{A})$ *contains a finite* **NP***-complete algebra then* $\mathbf{A}$ *is* **NP***-complete.*

Proof. We know $(\mathsf{HSP}(\mathbf{A}))_{fin} = \mathsf{HSP}_{fin}(\mathbf{A})$.

Let $\mathbf{B} = (B, F_B)$ be a subalgebra or a homomorphic image or a finite direct power of $\mathbf{A} = (A, F_A)$.

Take a finite $\Gamma \subseteq \mathrm{Inv}(F_B)$. We need to reduce $\mathrm{CSP}(\Gamma)$ to $\mathrm{CSP}(\Gamma')$ for some finite $\Gamma' \subseteq \mathrm{Inv}(F_A)$.

If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ then $\mathrm{Inv}(F_B) \subseteq \mathrm{Inv}(F_A)$, so we can take $\Gamma' = \Gamma$.

Proof: Homomorphic Images

Let $\psi : \mathbf{A} \twoheadrightarrow \mathbf{B}$ be a surjective homomorphism.

For a k-ary relation R on B, let

$$\psi^{-1}(R) = \{(a_1, \ldots, a_k) \in A^k \mid (\psi(a_1), \ldots, \psi(a_k)) \in R\}$$

Fact. If $R \in \mathrm{Inv}(F_B)$ then $\psi^{-1}(R) \in \mathrm{Inv}(F_A)$.

Take $\Gamma' = \{\psi^{-1}(R) \mid R \in \Gamma\}$.

The reduction from $\mathrm{CSP}(\Gamma)$ to $\mathrm{CSP}(\Gamma')$ is straightforward: an instance $\bigwedge R_i(\overline{s}_i)$ is transformed into $\bigwedge \psi^{-1}(R_i)(\overline{s}_i)$.

Proof: Finite Direct Powers

Let $\mathbf{B} = \mathbf{A}^k$.

Let R be an m-ary relation on A^k. Form an km-ary relation R' on A as follows: if $((a_{11}, \ldots, a_{1k}), \ldots, (a_{m1}, \ldots, a_{mk})) \in R$ then $(a_{11}, \ldots, a_{1k}, \ldots, a_{m1}, \ldots, a_{mk}) \in R'$.

Take $\Gamma' = \{R' \mid R \in \Gamma\}$. We have $\Gamma' \subseteq \mathrm{Inv}(F_A)$.

Take instance $\bigwedge R_i(x_1, \ldots, x_{n_i})$ of CSP(Γ). For every variable x_i in it, introduce new variables $x_i^1, \ldots, x_i^k$.

Transform the instance into an equivalent instance

$$\bigwedge R_i'(x_1^1, \ldots, x_1^k, \ldots, x_{n_i}^1, \ldots, x_{n_i}^k).$$

Varieties and Identities

Definition 8 *An equational class is a class of all algebras (in a given signature) satisfying a given set of identities.*

Example 1

- *Mal'tsev* $f(x,y,y) = f(y,y,x) = x$
- *Semilattice* $x \cdot x = x$, $x \cdot y = y \cdot x$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- *Near-unanimity (NU)*

$$f(y,x,\ldots,x) = f(x,y,\ldots,x) = \ldots = f(x,x,\ldots,y) = x$$

Theorem 8 (Birkhoff) *Varieties = equational classes.*

Thus, identities of $\mathbf{A}$ determine the complexity of CSP($\mathbf{A}$).

Idempotent Algebras

We have shown that we only need to consider constraint languages Γ which contain all constant relations $\{a\}$.

Then all polymorphisms of Γ are idempotent, that is, they satisfy the identity $f(x,\ldots,x) = x$.

Hence, we need to classify only idempotent algebras and idempotent varieties.

NP-complete Algebras: G-sets

For a permutation group G on D, a G-set is an algebra all whose operations are of the form $f(x_1, \ldots, x_n) = g(x_i)$ for some $g \in G$ and $1 \leq i \leq n$.
If a G-set is idempotent then $g = id$ and f is a projection.

Lemma 1 *If $\mathbf{A} = (D, F)$ is a non-trivial idempotent G-set then $\mathbf{A}$ is* **NP***-complete.*

Proof. Assume $0, 1 \in D$. $\mathrm{Inv}(F)$ is the set of all relations on D. Hence $R = \{0,1\}^3 \setminus \{(0,0,0),(1,1,1)\} \in \mathrm{Inv}(F)$.
Recall that CSP($\{R\}$) is the Not-All-Equal Sat problem, it's **NP**-complete.

NP-complete Algebras and Conjecture

Theorem 9 (Bulatov, Jeavons, K, 2005)
An idempotent algebra $\mathbf{A}$ *is* $\mathbf{NP}$*-complete if* $\mathbf{var}(\mathbf{A})$ *contains a* G*-set.*

Proposition 1 *For an idempotent algebra* $\mathbf{A}$, $\mathbf{var}(\mathbf{A})$ *contains a* G*-set iff* $\mathsf{HS}(\mathbf{A})$ *contains a* G*-set.*

All known $\mathbf{NP}$-complete algebras satisfy this condition.

Conjecture 2 (BJK, 2005) *(Structure of Dichotomy)*
An idempotent algebra $\mathbf{A}$ *is* $\mathbf{NP}$*-complete if* $\mathsf{HS}(\mathbf{A})$ *contains a* G*-set, and it is (globally) tractable otherwise.*

Graph H-colouring

For a digraph $H = (D, E)$, the H-COLOURING problem is whether a given digraph G admits a homomorphism to H. Note that H-COLOURING $=$ CSP$(\{E\})$.

Theorem 10 (Feder, Vardi, 1998) *For every finite Γ, there exists a digraph H_Γ such that the problems* CSP(Γ) *and H_Γ-*COLOURING *are polynomial-time equivalent.*

Theorem 11 (Hell, Nešetřil, 1990) *For an undirected loopless graph H, the H-*COLOURING *problem is tractable if H is bipartite and* **NP**-*complete otherwise.*

Graph H-colouring, cont'd

Assume $H = (D, E)$ is an undirected graph and all unary operations in $\text{Pol}(E)$ are permutations (i.e., H is a core).

Let $\Gamma = E \cup \{\{a\} \mid a \in D\}$, and let $\mathbf{A}_H = (D, \text{Pol}(\Gamma))$.

If H is bipartite then $D = \{a, b\}$ and $E = \{(a, b), (b, a)\}$.
It is easy to see that $\mathsf{HS}(\mathbf{A}_H)$ does not contain a G-set.

Theorem 12 (Bulatov, 2005)
If H is non-bipartite then $\mathsf{HS}(\mathbf{A}_H)$ contains a G-set.

Thus, the Structure of Dichotomy conjecture holds for undirected H-COLOURING problems.

Taylor Operations

Theorem 13 (Taylor, 1977)
For any finite idempotent algebra $\mathbf{A}$, *TFAE*

1. *The variety* $\mathrm{var}(\mathbf{A})$ *does not contain a G-set.*
2. *The algebra* $\mathbf{A}$ *has an n-ary term operation f satisfying n identities of the form*

$$f(x_{i1},\ldots,x_{in}) = f(y_{i1},\ldots,y_{in}), \quad i = 1,\ldots,n$$

   where all $x_{ij}, y_{ij} \in \{x,y\}$ *and* $x_{ii} \neq y_{ii}$.

Hence, if the conjecture is true then the presence of a Taylor term determines tractability of an idempotent algebra.

Weak Near-Unanimity Operations

An idempotent operation is called weak NU operation if
$f(y, x, \ldots, x) = f(x, y, \ldots, x) = \ldots = f(x, x, \ldots, y)$.

Note: any WNU operation is a Taylor operation

Theorem 14 (Maróti, McKenzie, 2006)
Any algebra with a Taylor term also has a WNU term.

Hence, to prove the conjecture, it is sufficient to show that any algebra with a term WNU operation is tractable.

More on this in talks by McKenzie and Maróti

How To Prove Tractability

Currently, the two main methods are:

- via local consistency (k-minimality or Datalog)

  More on this in a talk by Zádori

- via small generating sets

  More on this in a talk by Marković

k-minimality

Definition 9 *An instance (V, D, C) is called k-minimal if*

- *For each subset $W \subseteq V$, $|W| \leq k$, there is a constraint $(\overline{s}_i, R_i) \in C$ such that $W \subseteq \overline{s}_i$, and*
- *if $(\overline{s}_i, R_i), (\overline{s}_j, R_j) \in C$ and $W \subseteq \overline{s}_i \cap \overline{s}_j$ has size $\leq k$ then $\mathrm{pr}_W R_i = \mathrm{pr}_W R_j$*

Lemma 2 *For fixed k, every instance $\mathcal{P} = (V, D, C)$ of $\mathrm{CSP}(\mathbf{A})$ can be converted in polynomial time to a k-minimal instance $\mathcal{P}' = (V, D, C')$ of $\mathrm{CSP}(\mathbf{A})$ such that $Sol(\mathcal{P}) = Sol(\mathcal{P}')$.*

Proof

1. For each $U \subseteq V$, $|U| = k$, add a constraint (U, D^k).
2. Until constraints remain unchanged do

   For each $W \subseteq V$ with $|W| \le k$

   for each pair $(\overline{s}_i, R_i), (\overline{s}_j, R_j)$ with $W \subseteq \overline{s}_i \cap \overline{s}_j$

   – remove from R_i and R_j all tuples $\overline{a}$ with $\mathrm{pr}_W\, \overline{a} \notin \mathrm{pr}_W\, R_i \cap \mathrm{pr}_W\, R_j$

Note: If $R_i, R_j \in \mathrm{Inv}(F_A)$ then
$\{\overline{a} \in R_i \mid \mathrm{pr}_W\, \overline{a} \in \mathrm{pr}_W\, R_i \cap \mathrm{pr}_W\, R_j\}$ is again in $\mathrm{Inv}(F_A)$.

Relational Width

If imposing k-minimality on $\mathcal{P}$ produces $(\bar{s}_i, \emptyset)$, then $\mathcal{P}$ has no solution. The opposite is not always true.

Definition 10 *A constraint language* Γ *(an algebra* $\mathbf{A}$*) is said to have relational width* k *if every* k*-minimal instance of* $\mathrm{CSP}(\Gamma)$ *(of* $\mathrm{CSP}(\mathbf{A})$*, resp.) without empty constraint has a solution.*

A constraint language Γ *(an algebra* $\mathbf{A}$*) has bounded relational width if it has relational width* k *for some* k*.*

Observation. If an algebra $\mathbf{A}$ has relational width k, then any finite algebra in $\mathrm{var}(\mathbf{A})$ has relational width k, and is globally tractable.

Semilattices

Theorem 15 (Feder, Vardi, 1998) *Any algebra* $\mathbf{A}$ *with a semilattice term operation has relational width 1.*

Proof. Let $\mathcal{P} = (V, C)$ be a 1-minimal instance of CSP($\mathbf{A}$). By 1-minimality, if $c_i = (\overline{s}_i, R_i) \in C$ and $v \in \overline{s}_i$ then $\mathrm{pr}_v\, R_i$ depends neither on c_i nor on position of v in c_i.

If $\overline{s}_i = (v_1, \ldots, v_l)$ then $\bigvee R_i = (\bigvee \mathrm{pr}_{v_1} R_i, \ldots, \bigvee \mathrm{pr}_{v_l} R_i)$.

Hence, $\varphi(v) = \bigvee \mathrm{pr}_v\, R_i$ is a solution to $\mathcal{P}$.

Theorem 16 (Bulatov, 2006) *If* $\mathbf{A}$ *has a 2-semilattice term operation then* $\mathbf{A}$ *has relational width 3.*

(Recall: 2-semilattice means $xx = x, xy = yx, (xy)y = xy$.)

Near-Unanimity Operations

Recall NU operation is defined by the following identities:

$$f(y,x,\ldots,x) = f(x,y,\ldots,x) = \ldots = f(x,x,\ldots,y) = x$$

Theorem 17 (Jeavons, Cohen, Cooper, 1998)
Any algebra $\mathbf{A}$ *with a* k*-ary NU term operation has relational width* k*.*

Used in proof: $(k-1)$-decomposability

Theorem 18 (Baker, Pixley, 1975) *Let an algebra* $\mathbf{A}$ *have a* k*-ary NU term operation. Then, for any subalgebra* $\mathbf{B}$ *of* $\mathbf{A}^n$ *and any* $\overline{a} \in A^n$*, we have* $\overline{a} \in \mathbf{B}$ *iff* $\mathrm{pr}_U\,\overline{a} \in \mathrm{pr}_U\,\mathbf{B}$ *for all* $U \subseteq \{1,\ldots,n\}$, $|U| = k-1$.

Proof: Extension Property

Let $\mathcal{P} = (V, D, C)$ be a k-minimal instance of CSP($\mathbf{A}$).

Definition 11 *Let $V' \subseteq V$. The subinstance induced by V', denoted $\mathcal{P}|_{V'} = (V', D, C|_{V'})$, is obtained as follows: For each constraint $(\overline{s}_i, R_i)$ in $\mathcal{P}$, let I be the list of indices from V' in $\overline{s}_i$. Then $(\overline{s}_i, R_i)$ is replaced by* $(\mathrm{pr}_I\,\overline{s}_i, \mathrm{pr}_I\,R_i)$

Claim. For each subset $V' \subseteq V$, and each $v \in V \setminus V'$, any solution to $\mathcal{P}|_{V'}$ can be extended to a solution to $\mathcal{P}|_{V' \cup \{v\}}$.

The claim implies the theorem (just keep on extending). Assume the claim is false and derive a contradiction with Baker-Pixley (find a non-decomposable subpower of $\mathbf{A}$).

Proof: Construction

By assumption, there exist $W \subseteq V$, $v \in V \setminus W$, and a solution φ to $\mathcal{P}|_W$ that does not extend to $\mathcal{P}|_{W\cup\{v\}}$.

Let $W = \{w_1, \ldots, w_j\}$. Note that $j \geq k$ (by k-minimality).

Let $(\overline{s}_1, R_1), \ldots, (\overline{s}_m, R_m)$ be the constraints in $\mathcal{P}|_{W\cup\{v\}}$.
For $1 \leq i \leq m$, let $W_i = \{w^i_1, \ldots, w^i_j\}$.

Construct a new instance $\mathcal{P}'$ over variables $\bigcup_{i=1}^m W_i \cup \{v\}$ with constraints $(\overline{s}^*_1, R_1), \ldots, (\overline{s}^*_m, R_m)$ where each $\overline{s}^*_i$ is obtained from $\overline{s}_i$ by replacing each $w \in W$ in $\overline{s}_i$ by w^i.

Construct a new instance $\mathcal{P}'$ over variables $\bigcup_{i=1}^{m} W_i \cup \{v\}$ with constraints $(\overline{s}_1^*, R_1), \ldots, (\overline{s}_m^*, R_m)$ where each $\overline{s}_i^*$ is obtained from $\overline{s}_i$ by replacing each $w \in W$ in $\overline{s}_i$ by w^i.

Define the $(j \cdot m)$-ary relation R and tuple $\overline{a}$ as follows:

$$R = \{(\sigma(w_1^1), \ldots, \sigma(w_j^1), \ldots, \sigma(w_1^m), \ldots, \sigma(w_j^m)) \mid \sigma \in Sol(\mathcal{P}')\}$$

$$\overline{a} = (\varphi(w_1), \ldots, \varphi(w_j), \ldots, \varphi(w_1), \ldots, \varphi(w_j)).$$

We have that R is a subpower of $\mathbf{A}$, and, moreover, we have that $\overline{a} \notin R$ (otherwise φ would extend to $\mathcal{P}|_{W \cup \{v\}}$).

However, k-minimality of $\mathcal{P}$ implies the following: for every subset $U \subseteq \{1, \ldots, j \cdot m\}$ with $|U| = k - 1$, we have $\mathrm{pr}_U\, \overline{a} \in \mathrm{pr}_U\, R$. A contradiction.

How Bounded is Relational Width?

There are examples of algebras (2-semilattices) that have

- relational width 1
- relational width 3, but not 2

The following width hierarchy problem is open.

Problem 1 *For $n \geq 4$, is there a constraint language that has relational width n, but not $n-1$?*

Is there a constraint language that has bounded relational width, but not relational width 3?

Mal'tsev Algebras

Recall: Mal'tsev identities $f(x, y, y) = f(y, y, x) = x$.

Theorem 19 (Bulatov, Dalmau, 2006)
Any algebra $\mathbf{A}$ *with a Mal'tsev term is globally tractable.*

In an instance $\mathcal{P} = \{(\overline{s}_1, R_1), \ldots, (\overline{s}_q, R_q)\}$ of CSP($\mathbf{A}$), each constraint $(\overline{s}_i, R_i)$ describes subalgebra $\mathbf{A}_i$ of $\mathbf{A}^V$

- Take a "small" generating set (sgs) S_0 for $\mathbf{A}^V$.
- Assume we have a sgs S_j for $\bigcap_{i=1}^{j} \mathbf{A}_i$, $0 \leq j \leq q-1$.
- From S_j and $(\overline{s}_{j+1}, R_{j+1})$, compute S_{j+1}, a sgs for $\bigcap_{i=1}^{j+1} \mathbf{A}_i$.
- Then S_q is a sgs for $Sol(\mathcal{P}) = \bigcap_{i=1}^{q} \mathbf{A}_i$.

The Boolean Case

Theorem 20 (Post, 1941)
Every two-element idempotent algebra $\mathbf{A}$ *either is a G-set or else has one of the following term operations:*

- *semilattice* — $\mathrm{CSP}(\mathbf{A}) \subseteq$ HORN-SAT
- *majority* — $\mathrm{CSP}(\mathbf{A}) \subseteq$ 2-SAT
- *affine* $(x + y + z)$ — $\mathrm{CSP}(\mathbf{A}) \subseteq$ LIN EQ'S

Corollary 2 (Schaefer, 1978) *For constraint languages on* $\{0, 1\}$*, the Structure of Dichotomy conjecture holds.*

Bulatov's Dichotomy Theorems

Theorem 21 *The Structure of Dichotomy conjecture holds for all three-element algebras.*

Definition 12 *An algebra is called conservative if every subset is a subalgebra.*

Theorem 22 *The Structure of Dichotomy conjecture holds for all conservative algebras.*

Theorem 23 *Each three-element and each conservative algebra of bounded relational width has relational width 3.*

Meta-Problem

Theorem 24 (Bulatov, Jeavons)

- *Deciding whether, for a given constraint language* Γ *on a given set* D*, the algebra* $(D, \mathrm{Pol}(\Gamma))$ *generates a variety with a* G*-set is* **NP***-complete.*
- *Same problem with bounded* $|D|$ *is tractable.*
- *Deciding whether a given idempotent finite algebra generates a variety with a* G*-set is tractable.*

The Counting CSP

$\sharp$CSP(Γ): given an instance $\mathcal{P}$ of CSP(Γ), find $|Sol(\mathcal{P})|$.

$\sharp\mathrm{CSP}(\mathbf{A})$ = finding $|\bigcap_{i=1}^{q} \mathbf{A}_i|$. Varieties can also be used.

Theorem 25 (Bulatov, Dalmau) *$\sharp\mathrm{CSP}(\{R\})$ is hard for any binary reflexive non-symmetric relation R.*

Theorem 26 (Hagemann,Mitschke, 1973) *TFAE*

1. *$\mathcal{V}$ has a Mal'tsev term $f(x,y,y) = f(y,y,x) = x$.*
2. *For every algebra in $\mathbf{B} \in \mathcal{V}$, every reflexive subalgebra R in $\mathbf{B}^2$ is symmetric.*

Corollary 3 (Bulatov, Dalmau) *If $\mathrm{Pol}(\Gamma)$ contains no Mal'tsev operation then $\sharp\mathrm{CSP}(\Gamma)$ is $\sharp\mathbf{P}$-complete.*

Other CSP-related Problems

- Quantified CSP, QCSP(Γ)

$$\text{is the formula } \mathcal{Q}_1 x_1 \ldots \mathcal{Q}_n x_n \bigwedge R_i(\overline{s}_i) \text{ true?}$$

Algebras apply, but not varieties

More on this in a talk by Chen

- Maximum CSP, Max CSP(Γ)

find $\varphi : V \to D$ satisfying max number of constraints

Algebras don't apply, but lattices play important role.

More on this in a talk by Kuivinen

Congruence vs. Subalgebra Conditions

Congruences of $\mathbf{A}$ = equivalence relations in $\mathrm{Inv}(F_A)$ = kernels of homomorphisms from $\mathbf{A}$

The mainstream of UA is the study of varieties, in which all (finite) algebras satisfy a given congruence condition.

More on this in a talk by Willard

Prototype Theorem 1 *For an algebra* $\mathbf{A}$*, all (finite) algebras in* $\mathrm{var}(\mathbf{A})$ *satisfy congruence condition* X *if and only if* $\mathbf{A}$ *has term(s) satisfying a set* Y *of identities.*

There seems to be a correlation between (tame) congruence conditions and the complexity of CSP. (See talk by Larose)

Needed for CSP: link Y with properties of subpowers of $\mathbf{A}$.