

Special Lagrangians of cohomogeneity one in the resolved conifold

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Abstract. In this paper, which is a natural continuation of [IM], we describe some special Lagrangians of cohomogeneity one in the resolved conifold. Our main result gives a foliation of the resolved conifold by T^2 -invariant special Lagrangians, where the generic leaf is topologically $T^2 \times \mathbb{R}$. We also obtain a family of $SO(3)$ -invariant special Lagrangians. These special Lagrangian families in both the deformed and the resolved conifold approach asymptotically the same special Lagrangian cones in the conifold.

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1 Introduction

The notion of a special Lagrangian submanifold was introduced in 1984 by Harvey and Lawson in their seminal paper [HL] on calibrated geometries. Special Lagrangian geometry in Calabi-Yau manifolds has become an important subject due to a phenomenon in physics known as mirror symmetry. In 1996, Strominger, Yau and Zaslow [SYZ] conjectured that a compact Calabi-Yau 3-fold and its mirror should be foliated by special Lagrangian 3-tori with possibly singular fibres and the fibrations are dual to each other. This conjecture proposes a way to construct the mirror of a compact Calabi-Yau manifold, by an appropriate compactification of the dual of the special Lagrangian fibration. Since compact Calabi-Yau metrics cannot be written explicitly and also because singularities play a fundamental role, physicists have always been interested in finding concrete Calabi-Yau metrics on non-compact manifolds. One of the earliest examples of a pair of mirror Calabi-Yau metrics was found by Candelas and de la Ossa [CO] in 1990. The two manifolds arise from perturbing a singular cone on $S^2 \times S^3$ and are respectively known as the deformed and the resolved conifold. In fact, there is a 1-parameter family of Calabi-Yau metrics that passes through the singular metric and transforms the deformed conifold into the resolved conifold. The deformed conifold is a (trivial) \mathbb{R}^3 -bundle over S^3 and the resolved conifold is a \mathbb{C}^2 -bundle over S^2 . In passing through the singularity the special Lagrangian S^3 in the deformed conifold is pinched to a point and reappears in the resolved conifold as a holomorphic $\mathbb{C}P^1$. This is known as the *conifold transition* (see [CO] and [STY]).

In our recent paper [IM] we obtained a description of cohomogeneity one special Lagrangians in the deformed conifold. In particular, we found that the deformed conifold is

foliated by special Lagrangians which are generically $T^2 \times \mathbb{R}$. In the present paper we investigate the analogous problem for the resolved conifold and exhibit a foliation of the resolved conifold by T^2 -invariant special Lagrangians, where the generic leaf is topologically $T^2 \times \mathbb{R}$. We also obtain an $SO(3)$ -invariant family of special Lagrangians. It turns out that in both cases, the special Lagrangian families found in the resolved conifold have the same asymptotics as the special Lagrangians that we described in the deformed conifold in [IM], i.e. both families asymptotically approach the same special Lagrangians cones in the conifold.

2 Special Lagrangian Geometry

In what follows, we will review some definitions and set up notation.

Definition 2.1 *A Calabi-Yau n -fold (M, J, ω, Ω) is a Kähler n -dimensional manifold (M, J, ω) with Ricci-flat Kähler metric g and a nonzero holomorphic section Ω which trivializes the canonical bundle K_M .*

Since the metric g is Ricci-flat, Ω is a parallel tensor with respect to the Levi-Civita connection ∇^g [J1]. By rescaling Ω , we can take it to be the holomorphic $(n, 0)$ -form that satisfies:

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}, \quad (2.1)$$

where ω is the Kähler form of g . The form Ω is called the *holomorphic volume form* of the Calabi-Yau manifold M . Such a holomorphic volume form Ω exists if and only if the canonical bundle K_M is trivial, which implies that the first Chern class $c_1(M) = 0$ in $H^2(M, \mathbb{Z})$. Also, the above definition is equivalent to the manifold (M, g) having holonomy $SU(n)$.

Definition 2.2 *Let M^n be an n -dimensional Calabi-Yau manifold and $L \subset M$ a real n -dimensional submanifold of M . Then L is a special Lagrangian submanifold of M if $\omega|_L \equiv 0$ (i.e. L is Lagrangian) and $\text{Im} \Omega|_L \equiv 0$. Equivalently, $\text{Re}(\Omega)|_L$ is the volume form on L with respect to the induced metric.*

One important property of the special Lagrangian submanifolds is that they are absolutely area minimizing in their homology class, so they are in particular *minimal* submanifolds (see [HL]).

An example of a Calabi-Yau manifold is \mathbb{C}^n , with coordinates (z_1, \dots, z_n) , endowed with the flat metric g , the Kähler form ω_0 and the holomorphic volume form $\Omega_0 = dz_1 \wedge \dots \wedge dz_n$. \mathbb{R}^n is a trivial example of a special Lagrangian submanifold in \mathbb{C}^n . The following standard examples of cohomogeneity one special Lagrangians in \mathbb{C}^n are due to Harvey-Lawson.

Example 1:

$$L_c = \{\lambda u \mid u \in S^{n-1} \subset \mathbb{R}^n, \lambda \in \mathbb{C}, \text{Im}(\lambda^n) = c\}, \quad (2.2)$$

are special Lagrangians submanifolds of \mathbb{C}^n , invariant under the action of $SO(n) \subset SU(n)$, where $c \in \mathbb{R}$ is a constant.

Remark: The variety L_0 is an union of n special Lagrangian n -planes and when $c \neq 0$, each component of L_c is diffeomorphic to $\mathbb{R} \times S^{n-1}$ and it is asymptotic to L_0 .

Example 2: The special Lagrangian submanifolds in \mathbb{C}^n , invariant under the action of the maximal torus $T^{n-1} = \{ \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) | \theta_1 + \dots + \theta_n = 0 \}$ of $SU(n)$ are given by the equations:

$$\begin{aligned} |z_1|^2 - |z_j|^2 &= c_j, \quad j = 2, 3, \dots, n \quad \text{and} \\ \text{Re}(z_1 z_2 \dots z_n) &= c_1, \quad \text{if } n \text{ even} \\ \text{Im}(z_1 z_2 \dots z_n) &= c_1, \quad \text{if } n \text{ odd} \end{aligned}$$

where c_1, c_2, \dots, c_n are any real constants.

Remark: These special Lagrangians are topologically $T^{n-1} \times \mathbb{R}$ and they are asymptotic to cones on two flat tori in S^{2n-1} obtained by setting all constants equal to 0.

3 The resolved conifold as a Calabi-Yau manifold

In [IM], we studied the cohomogeneity one special Lagrangian submanifolds in the deformed conifold T^*S^3 . In this paper we will answer the same question in the resolved conifold. Both manifolds are Calabi-Yau and asymptotic to the cone on $S^2 \times S^3$ with a Ricci flat metric, called the *conifold*. Following the original paper of Candelas-de la Ossa [CO], we will first give a description of the resolved conifold and its Calabi-Yau structure.

Let Q_0 be the singular quadric in \mathbb{C}^4 defined by the equation:

$$\sum_{i=0}^3 z_i^2 = 0 \tag{3.3}$$

This quadric, called the conifold, is a cone on $S^2 \times S^3$. The deformed conifold is obtained by perturbing the conifold equation to $\sum_{i=0}^3 z_i^2 = \epsilon^2$, with ϵ a positive constant, and it admits a Calabi-Yau structure.

The other way to deal with the singularity of the conifold is to resolve it. Making the linear change of variables:

$$\begin{pmatrix} X \\ Y \\ U \\ V \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -i & -1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{3.4}$$

transforms the quadratic form $\sum_{i=0}^3 z_i^2$ to $2(XY - UV)$ and the conifold equation (3.3) becomes:

$$XY - UV = 0 \tag{3.5}$$

We note that the matrix

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & -i & -1 \end{pmatrix} \quad (3.6)$$

is not in $SO(4, \mathbb{C})$ (because it transforms the standard quadratic form), but it is an element of $U(4)$, since $P^{-1} = P^* = \bar{P}^t$. This shows that the Euclidean length is still preserved:

$$r^2 = \sum_{i=0}^3 |z_i|^2 = |X|^2 + |Y|^2 + |U|^2 + |V|^2 \quad (3.7)$$

The $SO(4, \mathbb{C})$ action on the z variables is now conjugated to an action on the new variables via $g \mapsto \tilde{g} = PgP^{-1}$, i.e.,

$$\tilde{g} \begin{pmatrix} X \\ Y \\ U \\ V \end{pmatrix} = PgP^* \begin{pmatrix} X \\ Y \\ U \\ V \end{pmatrix}, \quad g \in SO(4, \mathbb{C}) \quad (3.8)$$

Let

$$W = \begin{pmatrix} X & U \\ V & Y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} z_0 - iz_1 & z_3 - iz_2 \\ -z_3 - iz_2 & z_0 + iz_1 \end{pmatrix} \quad (3.9)$$

To resolve the conifold, one has to replace the equation (3.5) by the pair of equations:

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (3.10)$$

where $[\lambda_1 : \lambda_2]$ are homogeneous coordinates on $\mathbb{C}P^1 = S^2$. In other words, the resolved conifold M is the complex 3-fold defined by:

$$\{(X, Y, U, V, [\lambda_1 : \lambda_2]) \in \mathbb{C}^4 \times \mathbb{C}P^1 \mid XY - UV = 0, X\lambda_1 + U\lambda_2 = 0, V\lambda_1 + Y\lambda_2 = 0\} \quad (3.11)$$

Each point $(X, Y, U, V) \in Q_0$, except the origin, determines a unique point

$$[\lambda_1 : \lambda_2] = [-U : X] = [-Y : V] \in \mathbb{C}P^1$$

and the singularity at the origin is replaced (blown-up) by a copy of $\mathbb{C}P^1$. Outside the origin, the resolved conifold is topologically the same as the conifold.

Remark: There is a dual way to resolve the singularity where we replace the equation (3.10) by its transpose

$$(\lambda_1 \ \lambda_2) \begin{pmatrix} X & U \\ V & Y \end{pmatrix} = (0 \ 0)$$

These are known as small resolutions of the conifold. To obtain the “full” resolution of the singularity, one replaces the origin with the set of all complex lines through the origin that lie on the cone and thus the singularity would be replaced by an $S^2 \times S^2$ rather than a single S^2 .

We will also use inhomogeneous coordinates $\lambda_+ = \frac{\lambda_2}{\lambda_1}$ in the coordinate patch H_+ where $\lambda_1 \neq 0$ and $\lambda_- = \frac{\lambda_1}{\lambda_2}$ in the coordinate patch H_- , where $\lambda_2 \neq 0$. In H_+ we can take (U, Y, λ_+) and in H_- we can take (X, V, λ_-) as coordinates for the conifold. On the intersection $H_+ \cap H_-$ we have:

$$(X, V, \lambda_-) = (-\lambda_+ U, -\lambda_+ Y, \lambda_+^{-1})$$

In [CO], Candelas and de la Ossa showed that the resolved conifold is in fact the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ bundle over $\mathbb{C}P^1 \simeq S^2$ with fibre \mathbb{C}^2 . The radius in the fibre is given by

$$r^2 = \text{tr}(W^*W) = (1 + |\lambda_+|^2)(|U|^2 + |Y|^2) = (1 + |\lambda_-|^2)(|X|^2 + |V|^2)$$

Asymptotically, as $r \rightarrow \infty$, the resolved conifold approaches the singular conifold Q_0 . The Ricci-flat metric on M found in [CO] is:

$$g_{rc} = F'_a(r^2) \text{tr}(dW^*dW) + F''_a(r^2) |\text{tr}(W^*dW)|^2 + 4a^2 g_{S^2} \quad (3.12)$$

where F_a is a function of r^2 satisfying an appropriate differential equation and g_{S^2} is the Fubini-Study metric on S^2 with area π . In the patch H_+ , g_{S^2} is given by:

$$g_{S^2} = \frac{|d\lambda_+|^2}{(1 + |\lambda_+|^2)^2}$$

The resolution parameter a measures the size of the bolt $\mathbb{C}P^1$ ($a = 0$ for the conifold).

The differential equation satisfied by the function F_a is given by imposing the Ricci-flat condition. It is convenient to write the equation in terms of $\gamma = r^2 F'_a$, where the derivative is with respect to r^2 . One gets:

$$\gamma' \gamma (\gamma + 4a^2) = \frac{2}{3} r^2$$

where $\gamma' = \frac{d\gamma}{dr^2}$. Integrating yields:

$$\gamma^3 + 6a^2 \gamma^2 - r^4 = 0$$

which gives the solution as in [CO]:

$$\gamma = -2a^2 + 4a^4 N^{-\frac{1}{3}} + N^{\frac{1}{3}}, \quad N(r) = \frac{1}{2}(r^4 - 16a^6 + \sqrt{r^8 - 32a^6 r^4})$$

The function $F'_a(r^2)$ is therefore given by:

$$F'_a(r^2) = r^{-2} \left(-2a^2 + 4a^4 N^{-\frac{1}{3}} + N^{\frac{1}{3}} \right) \quad \text{where } N(r) = \frac{1}{2}(r^4 - 16a^6 + \sqrt{r^8 - 32a^6 r^4}) \quad (3.13)$$

For the conifold Q_0 with $a = 0$, we have $F'_0 = r^{-\frac{2}{3}}$.

The Kähler form $\omega_{rc}(v, w) = g_{rc}(Jv, w)$ on the resolved conifold can be expressed as a sum of two terms $\omega_{rc} = d\alpha_{rc} + 4a^2\omega_{S^2}$, where the one-form α_{rc} (compare [IM]) is given by:

$$\alpha_{rc} = F'_a(r^2) \operatorname{Im} \operatorname{tr}(W^*dW) = F'_a(r^2) \operatorname{Im} (\bar{X}dX + \bar{Y}dY + \bar{U}dU + \bar{V}dV) \quad (3.14)$$

and ω_{S^2} is the standard Kähler form of area π on S^2 . It can be expressed as $\omega_{S^2} = d\alpha_{\pm}$ in the two patches H_{\pm} on S^2 where:

$$\alpha_{\pm} = \frac{1}{2} \operatorname{Im} \frac{\lambda_{\pm} d\bar{\lambda}_{\pm}}{1 + |\lambda_{\pm}|^2} \quad \text{on } H_{\pm}$$

One can see that α_{rc} is invariant under the action of $SO(4) = SO(4, \mathbb{R})$. To show that α_{\pm} is also $SO(4)$ -invariant, we now compute the action of $SO(4)$ on $\mathbb{C}P^1$. The following three matrices acting on the z -coordinates generate the $\mathfrak{so}(4)$ as a Lie algebra:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.15)$$

Conjugating by the change of coordinate matrix P (3.4), we obtain the action on the X, Y, U, V -coordinates:

$$\tilde{B}_1 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \tilde{A}_3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \quad (3.16)$$

In H_+ , $\lambda_+ = -\frac{X}{U} = -\frac{V}{Y}$ and so $d\lambda_+ = -\frac{dX}{U} + \frac{XdU}{U^2}$. The action of \tilde{B}_1 on λ_+ is given by evaluating $d\lambda_+$ on the vector field corresponding to \tilde{B}_1 . We obtain: $\tilde{B}_1.\lambda_+ = -i\lambda_+$. Similarly, on H_- we compute that $\tilde{B}_1.\lambda_- = i\lambda_-$. The action of \tilde{B}_2 is the same as for \tilde{B}_1 and the action of \tilde{A}_3 is given by:

$$\tilde{A}_3.\lambda_+ = \frac{1}{2}(1 + \lambda_+^2), \quad \tilde{A}_3.\lambda_- = -\frac{1}{2}(1 + \lambda_-^2)$$

A computation now shows that the form α_{\pm} is invariant under the action of these three generators and hence under the $SO(4)$ -action. Therefore, the Kähler form and the metric are $SO(4)$ -invariant.

The holomorphic volume 3-form on M has the form:

$$\Omega_{rc} = dU \wedge dY \wedge d\lambda_+ = dV \wedge dX \wedge d\lambda_- \quad (3.17)$$

in local coordinates and it is also invariant under the action of $SO(4)$.

We now want to describe cohomogeneity one special Lagrangian in the resolved conifold M . As in our previous paper [IM], the only subgroups of $SO(4)$ that we need to consider are the maximal torus T^2 and the subgroup $SO(3)$. We will use the same moment map techniques as in the previous paper. We recall the following proposition which holds in general for any Kähler manifold M .

Proposition 3.1 *Let $G \subset SO(4)$ be a connected Lie subgroup with Lie algebra \mathfrak{g} and moment map $\mu : M \rightarrow \mathfrak{g}^*$ where (M, ω_{rc}) is the resolved conifold and \mathcal{O} an orbit of G in M . Then the orbit is isotropic, i.e. $\omega_{rc}|_{\mathcal{O}} \equiv 0$ if and only if $\mathcal{O} \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$, where $Z(\mathfrak{g}^*)$ is the centre of \mathfrak{g} .*

4 T^2 -invariant special Lagrangians

Under the coordinate transform (3.8), the action of the maximal torus T^2 of $SO(4)$ on the z -coordinates, described by the matrices:

$$\left\{ \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 & 0 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 & 0 \\ 0 & 0 & \cos \theta_2 & -\sin \theta_2 \\ 0 & 0 & \sin \theta_2 & \cos \theta_2 \end{pmatrix} \right\}$$

correspond to the following matrices with respect to the (X, Y, U, V) -coordinates:

$$\left\{ \begin{pmatrix} e^{-i\theta_1} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{-i\theta_2} \end{pmatrix} \right\}$$

The T^2 -action on the patch H_+ with coordinates (U, Y, λ) is given by:

$$g.(U, Y, \lambda_+) = (e^{i\theta_2}U, e^{i\theta_1}Y, e^{-i(\theta_1+\theta_2)}\lambda_+)$$

and on the patch H_- with coordinates (X, V, μ) is given by:

$$g.(X, V, \lambda_-) = (e^{-i\theta_1}X, e^{-i\theta_2}V, e^{i(\theta_1+\theta_2)}\lambda_-)$$

The following result gives all the special Lagrangian 3-folds of the resolved conifold invariant under the action of T^2 .

Theorem 4.1 *The special Lagrangian submanifolds in the resolved conifold M with coordinates $(X, Y, U, V, \lambda_1, \lambda_2)$ described by equations (3.11) with the Calabi-Yau structure, which are invariant under the action of the maximal torus T^2 of $SO(4)$ are given by the equations:*

$$\begin{aligned} \frac{1}{2}F'_a(r^2)(|X|^2 - |Y|^2) + 4a^2 \frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2} &= c_1 \\ \frac{1}{2}F'_a(r^2)(|V|^2 - |U|^2) + 4a^2 \frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2} &= c_2 \\ \text{Im}(XY) &= c_3 \end{aligned} \tag{4.18}$$

where F'_a is given by (3.13) and c_1, c_2 and c_3 are real constants.

Proof: Using the two infinitesimal generators:

$$\tilde{B}_1 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

of the T^2 -action on the resolved conifold, the moment map is given by:

$$\begin{aligned} \mu : M &\rightarrow (\mathfrak{t}^2)^* \simeq \mathbb{R}^2, \\ \mu &= \left(\frac{1}{2} F'_a(r^2)(|X|^2 - |Y|^2) + 4a^2 \mu_{S^2}, \frac{1}{2} F'_a(r^2)(|V|^2 - |U|^2) + 4a^2 \mu_{S^2} \right) \end{aligned}$$

where $\mu_{S^2}(\lambda_1, \lambda_2) = \frac{|\lambda_2|^2}{|\lambda_1|^2 + |\lambda_2|^2}$ is the moment map of the standard S^1 -action on $\mathbb{C}P^1$.

Since $(\mathfrak{t}^2)^* = \mathbb{R}^2$ is abelian, it follows from Proposition 3.1 that any T^2 -invariant special Lagrangian 3-fold L in Q^3 lies in a level set $\mu^{-1}(c)$, where $c = (c_1, c_2) \in \mathbb{R}^2$. The first two equations enforce this condition and ensures that the submanifold is Lagrangian. In order to impose the special Lagrangian condition at a given point $p = (U, Y, \lambda_+)$, we compute Ω_{rc} on the coordinate patch H_+ with coordinates on the three tangent vectors $Y_1 = \tilde{B}_1 p = (0, iY, -i\lambda_+)$, $Y_2 = \tilde{B}_2 p = (iU, 0, -i\lambda_+)$ and $Y_3 = \dot{p} = (\dot{U}, \dot{Y}, \dot{\lambda}_+)$:

$$\begin{aligned} \Omega_{rc}(Y_1, Y_2, Y_3) &= (dU \wedge dY \wedge d\lambda_+)(Y_1, Y_2, Y_3) \\ &= \begin{vmatrix} 0 & iU & \dot{U} \\ iY & 0 & \dot{Y} \\ -i\lambda_+ & -i\lambda_+ & \dot{\lambda}_+ \end{vmatrix} \\ &= \dot{U} Y \lambda_+ + U \dot{Y} \lambda_+ + U Y \dot{\lambda}_+ = (UY \lambda_+). \end{aligned} \tag{4.19}$$

Integrating the condition $\text{Im } \Omega_{rc} = 0$, we obtain $\text{Im}(UY \lambda_+) = c$ for $c \in \mathbb{R}$. Using $\lambda_+ = -\frac{X}{U}$ we finally obtain the third equation of the theorem. Note that in the other coordinate patch H_- we will obtain $\text{Im}(VX \lambda_-) = c$ which is the same equation since $\lambda_- = -\frac{Y}{V}$. \square

Remark 1: Equations (4.18) are obviously T^2 -invariant and linearly independent at a generic point. The above family of T^2 -invariant special Lagrangian 3-folds foliates the resolved conifold $M = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The generic orbit is $T^2 \times \mathbb{R}$ where T^2 is an orbit of the maximal torus in $SO(4)$. For the special values $c_1 = c_2$, $c_3 = 0$, these special Lagrangians intersect the $\mathbb{C}P^1$ in circles.

Remark 2: Asymptotically, as $r^2 \rightarrow \infty$ or equivalently as $a \rightarrow 0$, these special Lagrangians approach the special Lagrangian cone on T^2 in the conifold described by the equations:

$$\begin{aligned} |X|^2 - |Y|^2 &= 0 \\ |V|^2 - |U|^2 &= 0 \\ \text{Im}(XY) &= 0 \end{aligned}$$

which is the same asymptotic cone that was described in [IM]. In z coordinates on the conifold, the equations become: $\text{Im}(z_0 \bar{z}_1) = \text{Im}(z_2 \bar{z}_3) = \text{Im}(z_0^2 + z_1^2) = 0$

5 $SO(3)$ -invariant special Lagrangians

Let $G = SO(3)$ be the subgroup of $SO(4)$ embedded as:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, A \in SO(3) \right\}$$

The $SO(3)$ -invariant special Lagrangian 3-folds in the resolved conifold M are characterized in the following theorem:

Theorem 5.1 *The $SO(3)$ -invariant special Lagrangian submanifolds in the resolved conifold M , with coordinates $(X, Y, U, V, \lambda_1, \lambda_2)$ satisfying equation (3.5) are given by:*

$$L_c = \{\tilde{g}.p \in M \mid g \in SO(3), p = (0, Y, 0, 0), Y \neq 0, \operatorname{Re}(Y^2) = c\}$$

where \tilde{g} is the conjugated action on the (X, Y, U, V) -coordinates.

Proof:

Let $A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ be the infinitesimal generators of $SO(3)$. Using the change of coordinate matrix P (3.4), the action on the (X, Y, U, V) -variables is given by:

$$\tilde{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \tilde{A}_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i & i \\ 0 & 0 & i & -i \\ -i & i & 0 & 0 \\ i & -i & 0 & 0 \end{pmatrix}, \tilde{A}_3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}$$

The moment map for these generators is $\mu : M \rightarrow \mathfrak{so}(3)^*$ given by:

$$\mu = \left(\frac{1}{2}(|U|^2 - |V|^2), \frac{1}{2} \operatorname{Im}((U - V)(\bar{Y} - \bar{X})), \frac{1}{2} \operatorname{Im}((Y - X)(\bar{U} + \bar{V})) \right) + \mu_{S^2}$$

where

$$\mu_{S^2}(\lambda_+) = \left(\frac{|\lambda_+|^2}{1 + |\lambda_+|^2}, -\frac{1}{2(1 + |\lambda_+|^2)} \operatorname{Re}(\lambda_+ - |\lambda_+|^2 \bar{\lambda}_+), -\frac{1}{2(1 + |\lambda_+|^2)} \operatorname{Im}(\lambda_+ + |\lambda_+|^2 \bar{\lambda}_+) \right)$$

on the patch coordinate patch H_+ . A similar expression can be computed on H_- .

Let $p = (X, Y, U, V) \in M$. Since $Z(\mathfrak{so}(3)^*) = 0$, we need to start at a point in $\mu^{-1}(0)$. We make the simplifying assumption that $U = V = 0$. Using the fact that $XY = UV$, we can assume that $X = 0$ or $Y = 0$. If we choose $Y = 0$, then the point $(X, 0, 0, 0) \notin \mu^{-1}(0)$ since $\mu(X, 0, 0, 0)$ has the first component equal to 1. Hence we must choose $p = (0, Y, 0, 0)$, where $Y \neq 0$. Since $\lambda_+ = -\frac{Y}{\bar{Y}} = 0$, $p \in \mu^{-1}(0)$. We now look for curves $Y(s)$ in the complex Y -plane, which after applying the $SO(3)$ -action give rise to special Lagrangians of the form $L = \tilde{g}.Y(s)$, $g \in SO(3)$.

The tangent plane at p to L is spanned by the vectors: $\{v_1 = \tilde{A}_2 p, v_2 = \tilde{A}_3 p, v_3 = \dot{p}\}$, where the dot denotes differentiation with respect to the parameter s . The generic orbit of the

$SO(3)$ -action is an S^2 , the three infinitesimal generators are linearly dependent (Note that $\tilde{A}_1 p = 0$). The tangent space of L at the point p is therefore spanned by the vectors:

$$\left\{ v_1 = \frac{iY}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, v_2 = \frac{Y}{2} \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ \dot{Y} \\ 0 \\ 0 \end{pmatrix} \right\}$$

L is invariant under the $SO(3)$ -action and $\omega_{rc}|_{L=0} = 0$, since L lies in the zero level set of the moment map. We now impose the special Lagrangian condition $\text{Im } \Omega_{rc}|_{L=0} = 0$. Working on the patch H_+ with coordinates (U, Y, λ_+) , the holomorphic volume form is given by equation (3.17):

$$\Omega_{rc} = dU \wedge dY \wedge d\lambda_+$$

Since $\lambda_+ - \lambda_- = -\frac{V}{Y}$, we can instead use coordinates (U, Y, V) on and the holomorphic volume form in these coordinates becomes:

$$\Omega_{rc} = dU \wedge dY \wedge d\lambda_+ = -dU \wedge dY \wedge \frac{VdY - YdV}{Y^2} = -\frac{1}{Y} dU \wedge dY \wedge dV$$

Therefore,

$$\Omega_{rc}(v_1, v_2, v_3) = -\frac{1}{Y} dU \wedge dY \wedge dV(v_1, v_2, v_3)$$

Computing yields:

$$-\frac{1}{Y} \begin{vmatrix} \frac{iY}{2} & -\frac{Y}{2} & 0 \\ 0 & 0 & \dot{Y} \\ -\frac{iY}{2} & -\frac{Y}{2} & 0 \end{vmatrix} = -\frac{i}{2} Y \dot{Y}$$

Integrating, the condition $\text{Im } \Omega_{rc} = 0$ becomes:

$$\text{Re}(Y^2) = c \tag{5.20}$$

where c is a real constant. Let $Y = u + iv$. Then equation (5.20), gives a family of hyperbolas in the (u, v) -plane:

$$u^2 - v^2 = c$$

The special Lagrangian L_c is topologically $S^2 \times \mathbb{R}$ and has two components, each asymptotic to the special Lagrangian cones on S^2 given by setting $c = 0$ (obtained from the lines $u + v = 0$ and $u - v = 0$) in the conifold. This case is reminiscent of the flat case studied in [HL], but of a different dimension.

Remark 1: L_c does not intersect the zero-section (the bolt) S^2 of the resolved conifold, since $r^2 = |Y|^2$ is preserved under the $SO(3)$.

Remark 2: Note that asymptotically, the $SO(3)$ -invariant special Lagrangian in the resolved conifold approach the same special Lagrangian cones in the conifold as the $SO(3)$ -invariant special Lagrangian in the deformed conifold that we found in our previous paper [IM].

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