Cohomogeneity One Special Lagrangian Submanifolds in the Deformed Conifold

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Abstract. In this paper we describe the cohomogeneity one special Lagrangian 3-folds in the cotangent bundle of the 3-sphere, also known in the physics literature as a deformed conifold. Our main result gives a global foliation of the deformed conifold by T^2-invariant special Lagrangian 3-folds, where the generic leaf is topologically $T^2 \times \mathbb{R}$. In the limit, these special Lagrangians asymptotically approach a special Lagrangian cone on a torus in the conifold. Using moment map techniques we also recover the family of $SO(n)$-invariant special Lagrangian n-folds in T^*S^n obtained by H. Anciaux in [An].

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1 Introduction

Beginning with the seminal paper by R.Harvey and H.B. Lawson [HL] on calibrated geometry, there has been extensive research in the mathematics literature on special Lagrangian and other calibrated submanifolds. Recently, a lot of progress has been done in constructing special Lagrangian submanifolds using various techniques. To give some examples, D. Joyce used the method of ruled submanifolds, integrable systems and evolution of quadrics in [J3, J4, J5] to construct explicit examples of special Lagrangian m-folds in $\mathbb{C}^m$, M. Haskins exhibited examples of special Lagrangian cones in $\mathbb{C}^3$ [Ha], etc. Although the main emphasis has been on examples in flat space, such as $\mathbb{C}^n$, there has also been progress in studying the special Lagrangian submanifolds in non-flat Calabi-Yau manifolds. R. Schoen and J. Wolfson used variational methods to construct special Lagrangians in Calabi-Yau manifolds in [SW]. H. Anciaux found new $SO(n)$-invariant examples in T^*S^n [An], equipped with the Ricci-flat Stenzel metric. See also [IKM] and [KM], where a different method is used to construct calibrated submanifolds.

In this paper, we will find new examples of cohomogeneity one special Lagrangians in the Calabi-Yau manifold T^*S^3, which is known in the physics literature as the deformed conifold. The main result is that we exhibit a foliation of T^*S^3 by special Lagrangians, where the generic leaf is $T^2 \times \mathbb{R}$ and the T^2 is an orbit under the maximal torus of the isometry group $SO(4)$. We will show that asymptotically these special Lagrangians converge to a special Lagrangian cone in the conifold, which is the cone on $S^2 \times S^3$, with the singular Calabi-Yau metric [CO]. We will also recover some of Anciaux's results about $SO(3)$-invariant special Lagrangians using different techniques. Our methods use the moment map of the $SO(4)$ action and are similar to Joyce's methods [J2] for the flat case.

The Calabi-Yau metric on T^*S^3, the deformed conifold, was first explicitly described by Candelas and de la Ossa [CO] in 1990, before Stenzel (independently) discovered it for any T^*S^n, and more generally, for any cotangent bundles of rank one symmetric spaces. The case T^*S^2 is already very interesting and the Ricci-flat hyperkähler metric on this manifold was first discovered by Eguchi and Hanson [EH] in 1978. This 4-dimensional metric is the basic model of a number of explicit special holonomy metrics discovered by physicists.

Special Lagrangian submanifolds in Calabi-Yau spaces play an important role in string theory, particularly in mirror symmetry. Since Calabi-Yau metrics are difficult to describe explicitly on compact manifolds, it is a good strategy to understand the local non-compact models first. Moreover, it is important to study how singularities develop, since they provide a transition between different topological types in the moduli space of Calabi-Yau structures. In dimension 6, the basic example is that of a conifold transition, where a Calabi-Yau metric on a smooth manifold diffeomorphic to T^*S^3, called the deformed conifold, degenerates into a singular cone metric on $S^2 \times S^3$ and then is resolved (by a blow-up) into another Calabi-Yau metric on the total space of a $\mathbb{C}^2$-bundle $(\mathcal{O}(-1) + \mathcal{O}(-1))$ on S^2, which is known as the *small resolution* of the conifold and is the mirror of the deformed conifold. This transition and the metrics involved are described explicitly by Candelas and de la Ossa [CO].

We now give a brief description of the contents of this paper. In section 2, we review the basic facts about Calabi-Yau manifolds and special Lagrangians, including a list of well-known relevant examples. In sections 3 we describe the Calabi-Yau structure on the deformed conifold and the Stenzel metric on T^*S^n. We define the associated moment map and discuss some of its basic properties in section 4. In section 5 we prove that the only homogenous example of special Lagrangians 3-fold in T^*S^3 is the zero section and in section 6 we compute our main examples of cohomogeneity one special Lagrangians including a description of the asymptotics. We conclude with some remarks and open questions in the last section.

2 Special Lagrangian Geometry

Special Lagrangian submanifolds are a special class of minimal submanifolds in Calabi-Yau spaces and were introduced by Harvey and Lawson in their seminal paper [HL] using the notion of a calibration. We begin by quickly reviewing the basic definitions and setting up notations. For details, see [J1].

2.1 Basic Definitions

Definition 2.1 *A Calabi-Yau n-fold (M, J, ω, Ω) is a Kähler n-dimensional manifold (M, J, ω) with Ricci-flat Kähler metric g and a nonzero holomorphic section Ω which trivializes the canonical bundle K_M.*

Since the metric g is Ricci-flat, Ω is a parallel tensor with respect to the Levi-Civita connection ∇^g [J1]. By rescaling Ω, we can take it to be the holomorphic $(n, 0)$-form that satisfies:

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}, \tag{2.1}$$

where ω is the Kähler form of g. The form Ω is called the *holomorphic volume form* of the Calabi-Yau manifold M.

Let φ be a closed p-form on manifold M. We say φ is a calibrating form on M if

$$\varphi_{|V} \leq vol_V$$

for any oriented p-plane $V \subset T_xM, \forall x \in M$. A submanifold N of M is called *calibrated by* φ if $\varphi\ |_{T_xN} = vol_{T_xN}, \forall x \in N$.

Remark: The constant factor in (2.1) is chosen so that $\mathrm{Re}\,\Omega$ becomes a calibration on M.

Definition 2.2 *Let (M, J, ω, Ω) be a Calabi-Yau n-fold and $L \subset M$ a real oriented n-dimensional submanifold of M. Then L is called a special Lagrangian submanifold of M if it is calibrated by* $\mathrm{Re}\,\Omega$.

More generally, L is said to be a special Lagrangian submanifold with phase $e^{i\theta}$ if it is calibrated by $\mathrm{Re}(e^{-i\theta}\Omega)$, where θ is a constant.

An alternative and very useful description of the special Lagrangian submanifolds found in [HL] is the following:

Proposition 2.3 *Let (M, J, ω, Ω) be an n-dimensional Calabi-Yau manifold and $L \subset M$ a real n-dimensional submanifold of M. Then L is called a special Lagrangian submanifold of M if $\omega\ |_L \equiv 0$ and $\mathrm{Im}\,\Omega\ |_L \equiv 0$.*

Remark: The condition $\omega\ |_L \equiv 0$ says that L is a Lagrangian submanifold. Therefore, special Lagrangian submanifolds are Lagrangian with the extra condition $\mathrm{Im}\,\Omega\ |_L \equiv 0$.

The simplest example of a Calabi-Yau manifold is $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$, endowed with the flat metric $<,>$, the Kähler form ω_0 and the holomorphic volume form Ω_0, where:

$$< z, w > = \mathrm{Re} \sum_{i=1}^{n} z_i \bar{w}_i \tag{2.2}$$

$$\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n) \tag{2.3}$$

$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_n \tag{2.4}$$

$\mathbb{R}^n$ is a trivial example of a special Lagrangian submanifold in $\mathbb{C}^n$.

One important property of the special Lagrangian submanifolds is that they are absolutely area minimizing in their homology class, so they are in particular *minimal* submanifolds (see [HL]).

The graph of $F : \mathbb{R}^n \to \mathbb{R}^n$ is a Lagrangian n-fold in $\mathbb{R}^{2n}$ if and only if $F = \nabla f$, for some function $f : \mathbb{R}^n \to \mathbb{R}$. The graph is special Lagrangian if and only if the function f satisfies the differential equation:

$$\mathrm{Im}\, det_{\mathbb{C}}(I + i\mathrm{Hess}\ f) = 0 \text{ on } \mathbb{C}^n$$

This is a nonlinear elliptic P.D.E. and it is difficult to solve in general. One idea, initiated by Harvey and Lawson, is to look for solutions invariant under certain group actions of $\mathbb{C}^n$. In their paper [HL], Harvey and Lawson produced many examples of symmetric special Lagrangian. We will review some of their examples relevant to this paper in the next section.

2.2 Cohomogeneity one examples in $\mathbb{C}^n$

Let L be a special Lagrangian submanifold of $\mathbb{C}^n$. The group of automorphisms preserving the Calabi-Yau structure on $\mathbb{C}^n$ is $G = SU(n) \ltimes \mathbb{C}^n$. The symmetry group of L is defined to be the Lie subgroup of $SU(n) \ltimes \mathbb{C}^n$ that acts on $\mathbb{C}^n$ leaving L fixed. In general, the easiest special Lagrangian submanifolds to construct are those with large symmetry groups. It can be shown that all homogeneous special Lagrangian submanifolds of $\mathbb{C}^n$ are conjugate under $SU(n) \ltimes \mathbb{C}^n$ to the standard real n-plane $\mathbb{R}^n \subset \mathbb{C}^n$ [J1]. The next most symmetric special Lagrangians are those of cohomogeneity one, i.e. the ones whose orbits of the symmetry group are of codimension one in L. We will now recall two important examples of cohomogeneity one special Lagrangian submanifolds found by Harvey and Lawson in [HL].

Example 1: The following are special Lagrangian submanifolds of $\mathbb{C}^n$, invariant under the action of $SO(n) \subset SU(n)$:

$$L_c = \{\lambda\, u \mid u \in S^{n-1} \subset \mathbb{R}^n,\ \lambda \in \mathbb{C},\ \mathrm{Im}(\lambda^n) = c\,\}, \tag{2.5}$$

where $c \in \mathbb{R}$ is a constant. The variety L_0 is an union of n special Lagrangian n-planes and when $c \neq 0$, each component of L_c is diffeomorphic to $\mathbb{R} \times S^{n-1}$ and it is asymptotic to L_0.

Example 2: The special Lagrangian submanifolds in $\mathbb{C}^n$, invariant under the action of the maximal torus $T^{n-1} = \{\ \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n}) | \theta_1 + \ldots \theta_n = 0\}$ of $SU(n)$ are given by the equations:

$$\begin{aligned} &|z_1|^2 - |z_j|^2 = c_j,\ j = 2, 3, \ldots, n \quad \text{and} \\ &\mathrm{Re}(z_1 z_2 \ldots z_n) = c_1,\ \text{ if } n \text{ even} \\ &\mathrm{Im}(z_1 z_2 \ldots z_n) = c_1,\ \text{ if } n \text{ odd} \end{aligned}$$

where $c_1, c_2, \ldots c_n$ are any real constants.

Remark: These special Lagrangians are topologically $T^{n-1} \times \mathbb{R}$ and they are asymptotic to the union of two cones on flat tori T^{n-1} in S^{2n-1} obtained by making all constants equal to 0.

Using moment map techniques, D. Joyce [J1] made a systematic study of cohomogeneity one special Lagrangian folds in $\mathbb{C}^3$ and he found the following new example:

Example 3: The submanifold given by

$$L_{a,b,c} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - |z_2|^2 = a,\ \mathrm{Re}(z_1 z_2) = b,\ \mathrm{Im}(z_3) = c\}$$

where a, b, c are any real constants is a special Lagrangian in $\mathbb{C}^3$ and it is invariant under the subgroup $U(1) \ltimes \mathbb{R}$ of $SU(3) \ltimes \mathbb{C}^3$, acting by

$$(e^{i\theta}, x).(z_1, z_2, z_3) \rightarrow (e^{i\theta} z_1, e^{-i\theta} z_2, x + z_3)$$

for $\theta \in [0, 2\pi), x \in \mathbb{R}$.

D. Joyce in [J2] proved that the above examples are all the cohomogeneity one special Lagrangian submanifolds in $\mathbb{C}^3$:

Theorem 2.4 *Every cohomogeneity one special Lagrangian 3-fold in $\mathbb{C}^3$ is conjugate under $SU(3) \ltimes \mathbb{C}^3$ to a subset of $\mathbb{R}^3$, one of the 3-folds of Example 1 and Example 2 with $n = 3$ or one of those in Example 3.*

3 T^*S^n as a Calabi-Yau manifold

In 1995, M. Stenzel [St] discovered that the cotangent bundle of the sphere, T^*S^n, and more generally the cotangent bundle of symmetric spaces of rank one, can be endowed with a Ricci-flat metric making them into Calabi-Yau manifolds. As mentioned in the introduction, the lower dimensional cases were discovered in the physics literature. The case $n = 2$ is the Eguchi-Hanson metric [EH] and the case $n = 3$ (the deformed conifold) is due to Candelas and de la Ossa [CO].

In what follows we will first describe the cotangent bundle of the sphere as a complex affine quadric following Szöke [Sz] and define the Kähler potential of the Stenzel metric. For more details on this material, the reader should consult [St].

3.1 The Stenzel metric

Let $T^*S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1},\ |x| = 1,\ x \cdot \xi = 0\}$ be the cotangent bundle of the n-sphere (which we have identified with the tangent bundle). The group $SO(n+1, \mathbb{R})$ acts with cohomogeneity one on T^*S^n, the generic orbit being $SO(n+1)/SO(n-1)$ (where $|\xi|$ is constant). According to Szöke [Sz], one can identify T^*S^n with the affine quadric

$$Q^n = \{z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} |\ \sum_{i=0}^{n} z_i^2 = 1\}$$

using the diffeomorphism $h : T^*S^n \to Q^n$ given by:

$$(x, \xi) \to z = x \cosh(|\xi|) + i\ \frac{\sinh(|\xi|)}{|\xi|} \xi \tag{3.6}$$

This diffeomorphism is equivariant with respect to the action of $SO(n+1, \mathbb{R})$ on T^*S^n and the natural action of $SO(n+1, \mathbb{C})$ on Q^n. The complex structure on the cotangent bundle of the n-sphere is obtained by pulling back the complex structure of the affine quadric under the map h. On the complex quadric, there exists a Ricci-flat metric whose corresponding symplectic form is the Stenzel form given by:

$$\omega_{St} = i\partial\bar{\partial}u(r^2) = i\sum_{j=0}^{n}\sum_{k=0}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_k} u(r^2) dz_j \wedge d\bar{z}_k$$

where $r^2 = |z|^2 = \sum_{j=0}^{n} z_j \bar{z}_j = \cosh(2|\xi|)$ and $u(r^2)$ is a smooth real function satisfying the following differential equation:

$$\frac{d}{d\tau}(u'(\tau))^n = c\, n\, (\sinh \tau)^{n-1} \tag{3.7}$$

where $\tau = \cosh^{-1}(r^2)$ and c is a positive constant (see [St]). In dimension $n = 2$, there is an explicit formula for the potential function: $u(r^2) = \sqrt{1 + r^2}$. In dimension $n = 3$, the derivative of the potential function is given by the relation:

$$u'(\tau)^3 = \frac{3c}{2}\left(\frac{\sinh(2\tau)}{2} - \tau\right) \tag{3.8}$$

which using the initial condition $u'(0) = 0$ integrates to the more complicated formula:

$$u(r^2) = \int_0^{\cosh^{-1}(r^2)} \left[\frac{3c}{2}\left(\frac{\sinh(2\sigma) - \sigma}{2}\right)\right]^{\frac{1}{3}} d\sigma \tag{3.9}$$

The form $\omega_{St} = i\partial\bar{\partial}u$ is exact on T^*S^n and $\omega_{St} = d\alpha_{St}$, where $\alpha_{St} = -\operatorname{Im}(\bar{\partial}u)$. The form α_{St} is related to the Liouville form $\alpha_0(v) = \frac{1}{2} < v, Jz >$ on $\mathbb{C}^{n+1}$ by $\alpha_{St} = u'(|z|^2)\alpha_0$. Therefore, it follows that the 1-form α_{St} has the expression:

$$\alpha_{St}(v) = \frac{1}{2}u'(|z|^2)\omega_0(z, v) = \frac{1}{2}u'(|z|^2) < v, Jz >, \ v \in T_zQ, z \in Q \tag{3.10}$$

where $<, >$ and ω_0 are respectively the flat Euclidean metric and the Kähler form of $\mathbb{C}^{n+1}$. It is well known that on the complex space $\mathbb{C}^{n+1}$, $\omega_0(v, w) =< Jv, w >$, where J is the complex structure on $\mathbb{C}^{n+1}$.

Differentiating expression (3.10), we calculate:

$$\begin{aligned}
\omega_{St}(v, w) &= d\alpha_{St}(v, w) = v(\alpha_{St}(w)) - w(\alpha_{St}(v)) - \alpha_{St}([v, w]) \\
&= v\left(\frac{1}{2}u'(|z|^2) < w, Jz >\right) - w\left(\frac{1}{2}u'(|z|^2) < v, Jz >\right) - \alpha_{St}([v, w]) \\
&= \frac{1}{2}\{u''(|z|^2)v(|z|^2) < w, Jz > + u'(|z|^2)v(< w, Jz >) - u''(|z|^2)w(|z|^2) < v, Jz > \\
&- u'(|z|^2)w(< v, Jz >) - u'(|z|^2) < [v, w], Jz >\} \\
&= u'(|z|^2) < w, Jv > + u''(|z|^2)(< v, z >< w, Jz > - < w, z >< v, Jz >) \\
&= u'(|z|^2)\omega_0(v, w) + u''(|z|^2)(< v, z > \omega_0(z, w) - < w, z > \omega_0(z, v))
\end{aligned}$$

In the above calculation we used that $v(|z|^2) = 2 < v, z >$, $\nabla_v w - \nabla_w v = [v, w]$ and the fact that $< v, Jw >= - < w, Jv >$.

Therefore, the Kähler form of the Stenzel metric at a point z on the quadric Q is given by:

$$\omega_{St}(v, w) = u'(|z|^2)\,\omega_0(v, w) + u''(|z|^2)\Big(< w, z > \omega_0(v, z) - < v, z > \omega_0(w, z)\Big), \ v, w \in T_zQ \tag{3.11}$$

Formulas (3.10) and (3.11) will prove to be important when we compute the moment maps for group actions on the quadric.

On the quadric Q define the holomorphic $(n, 0)$-form Ω_{St} by the relation:

$$\frac{1}{2}d(z_0^2 + z_1^2 + \cdots + z_n^2 - 1) \wedge \Omega_{St} = \Omega_0$$

where $\Omega_0 = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_n$ is the holomorphic volume form of $\mathbb{C}^{n+1}$. Therefore:

$$\Omega_{St}(v_1, v_2, \ldots, v_n) = \Omega_0\left(z, v_1, \ldots, v_n\right), \; v_1, \ldots, v_n \in T_z Q, \; z \in Q \tag{3.12}$$

The quadric Q^n with this structure becomes a Calabi-Yau manifold since equation (2.1) holds for ω_{St} and the corresponding holomorphic n-form Ω_{St}, up to a multiplicative constant (see also [An]).

3.2 The conifold in dimension 3

Let Q_0 be the quadric in $\mathbb{C}^4$ defined by the equation:

$$\sum_{i=0}^{3} z_i^2 = 0$$

This quadric, called the conifold, is singular at the origin and represents a cone on $T_1(S^3) \cong S^2 \times S^3$. The complex structure of the conifold is given by the embedding

$$h_0 : T_1(S^3) = \left\{(x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^4 | \; |x| = |\xi| = 1, \; x \cdot \xi = 0\right\} \to Q_0, \; z = h_0(x, \xi) = x + i\xi$$

Deforming the conifold equation to $\sum_{i=0}^{3} z_i^2 = \epsilon^2$, with ϵ a positive constant, yields the complex quadric Q_ϵ in dimension 3, where ϵ is the radius of the zero section S^3. This is equivalent to replacing the tip of the conifold by an S^3(see [CO]) and as $\epsilon \to 0$, this sphere collapses into the singular point of Q_0. In the physical literature, the complex quadric Q_ϵ is also known as a *deformed conifold*.
Candelas and de la Ossa [CO] showed that the conifold Q_0 admits a Ricci-flat metric g_{cone} with Kähler potential $u_{cone}(r^2) = \frac{3}{2} r^{\frac{4}{3}}$. The holomorphic $(n, 0)$-form Ω_{cone} defined by:

$$\frac{1}{2} d(z_0^2 + z_1^2 + \cdots + z_3^2) \wedge \Omega_{cone} = dz_0 \wedge dz_1 \wedge \cdots \wedge dz_3 \tag{3.13}$$

makes the conifold into a singular Calabi-Yau manifold.

We note that the above relation can be used to compute Ω_{cone} as:

$$\Omega_{cone}(v_1, v_2, v_3) = \frac{1}{|z|^2} (dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(\bar{z}, v_1, v_2, v_3) \tag{3.14}$$

where $v_1, v_2, v_3 \in T_z Q_0$, $z \in Q_0$ and $\bar{z} = \bar{z}_0 \frac{\partial}{\partial z_0} + \bar{z}_1 \frac{\partial}{\partial z_1} + \bar{z}_2 \frac{\partial}{\partial z_2} + \bar{z}_3 \frac{\partial}{\partial z_3}$.
We make the remark that we can not use the position vector z anymore on the cone in this case to calculate Ω_{cone} in terms of Ω_0, since $d(z_0^2 + z_1^2 + z_2^2 + z_3^2) \wedge \Omega_{cone}(z, v_1, v_2, v_3) = 0$. Instead, we use the vector $\bar{z}$ which is normal to the cone and the fact that $\frac{1}{2} d(z_0^2 + z_1^2 + \cdots + z_3^2)(\bar{z}) = |z|^2$.

4 Moment Maps and Special Lagrangians with Symmetry

The group $SO(n+1, \mathbb{R})$ acts on the cotangent bundle of the sphere

$$T^* S^n = \{(x, \xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \; |x| = 1, \; x \cdot \xi = 0\} \tag{4.15}$$

with cohomogeneity one. The action is transitive on the sets $|\xi| = \rho =$ constant and is given by:

$$g.(x, \xi) = (gx, g\xi), \; g \in SO(n+1), \; (x, \xi) \in T^*S^n \tag{4.16}$$

We would like to find examples of special Lagrangian submanifolds in the deformed conifold T^*S^3, with large symmetry group. Our method will use moment map techniques. In what follows we will compute the moment map of a group action on the cotangent bundle of the sphere and review some results of Joyce [J2] about finding G-invariant special Lagrangian submanifolds in $\mathbb{C}^n$ by using the moment map. These results are quite general and easily extend from the case of the flat $\mathbb{C}^n$ to general symplectic manifolds with Hamiltonian actions and in particular to the Calabi-Yau manifold T^*S^n.

Let $Q \cong T^*S^n$ be endowed with the Calabi-Yau structure described in section 3.1. As we have seen, the group of automorphisms of T^*S^n preserving the Calabi-Yau structure is $SO(n+1, \mathbb{R}) \subset SO(n+1, \mathbb{C})$. Let G be a Lie subgroup of $SO(n+1, \mathbb{R})$, with Lie algebra $\mathfrak{g}$. Let $A \in \mathfrak{g} \subset \mathfrak{o}(n+1)$. Then the induced vector field on Q is given by: $z \mapsto X_A(z) = Az$ with flow $z \mapsto e^{tA}z$ for $z \in Q$. When the symplectic form is exact, like in the case of the Stenzel form $\omega_{St} = d\alpha_{St}$, the moment map $\mu : T^*S^n \to \mathfrak{g}^*$ of the G-action can be computed easily as follows:
Both forms ω_{St} and α_{St} are invariant under the flow of X_A, that is

$$\mathcal{L}_{X_A}\omega_{St} = \mathcal{L}_{X_A}\alpha_{St} = 0$$

for any $A \in \mathfrak{g} \subset \mathfrak{o}(n+1)$.
From Cartan's formula, $\mathcal{L}_{X_A}\alpha_{St} = d(X_A \lrcorner \alpha_{St}) + X_A \lrcorner d\alpha_{St}$ and using $d\alpha_{St} = \omega_{St}$, we see that

$$-X_A \lrcorner \omega_{St} = d(\alpha_{St}(X_A))$$

This shows that the action of G on T^*S^n is hamiltonian, with moment map $\mu(z)$ given by $A \mapsto \alpha_{St}(A)$ which can be rewritten as:

$$< \mu(z), A >= \mu_A(z) = \alpha_{St}(X_A(z))$$

where $z \in Q \subset \mathbb{C}^{n+1}$, $A \in \mathfrak{g} \subset \mathfrak{o}(n+1)$ and $<,>$ is the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Using formula (3.10) for α_{St}, we have thus proved the following proposition.

Proposition 4.1 *Let $G \subset SO(n+1)$ be a connected Lie group. The moment map of the G-action on the complex quadric Q is given by:*

$$\mu_A : Q^n \to \mathbb{R}, \quad \mu_A(z) = \alpha_{St}(Az) = \frac{1}{2}u'(|z|^2) < Az, Jz >, \; \forall A \in \mathfrak{g} \tag{4.17}$$

where $<,>$ is the Euclidean metric on $\mathbb{C}^{n+1}$ and the function u satisfies equation (3.7).

We define the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ to be the subspace of $\mathfrak{g}$ fixed by the coadjoint action of G. Since the moment map is equivariant, a level set of the moment map $\mu^{-1}(c)$ for $c \in \mathfrak{g}^*$ is G-invariant if and only if $c \in Z(\mathfrak{g}^*)$. In order to find examples of special Lagrangian submanifolds in Q, we will use the following proposition which is true for moment maps on any symplectic manifold, in particular on Q.

Proposition 4.2 *Let $G \subset SO(n+1)$ be a connected Lie subgroup with Lie algebra $\mathfrak{g}$ and moment map $\mu : T^*S^n \to \mathfrak{g}^*$ and $\mathcal{O}$ an orbit of G in T^*S^n. Then the orbit is isotropic, i.e. $\omega_{St}|_{\mathcal{O}} \equiv 0$ if and only if $\mathcal{O} \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$.*

Proof: Let $v, w \in \mathfrak{g}$ and X_v, Y_w be the induced vector fields on the manifold T^*S^n, i.e. the flow of X_v is $m \in T^*S^n \to e^{vt}.\, m$ and similarly, the flow of Y_w is $m \in T^*S^n \to e^{wt}.\, m$. Then:

$$\omega_{St}(X_v, Y_w) = (X_v \lrcorner \omega_{St})(Y_w) = d\mu_v(Y_w) = \mathcal{L}_{Y_w}(\mu_v) \tag{4.18}$$

Since $\mathcal{O}$ is a G-orbit, the vector fields X_v for any $v \in \mathfrak{g}$ generate all the tangent space of the orbit. So, $\mathcal{O}$ is isotropic if and only if $\omega(X_v, Y_w) = 0$ for any $v, w \in \mathfrak{g}$. Equation 4.18 implies that $\mathcal{L}_{Y_w}\mu = 0$ on $\mathcal{O}, \forall w \in \mathfrak{g}$ which means that the moment map μ is constant on $\mathcal{O}$, i.e. $\mathcal{O} \subseteq \mu^{-1}(c)$, for some $c \in \mathfrak{g}^*$. Since $\mathcal{O}$ is G-invariant, $c \in Z(\mathfrak{g}^*)$. $\square$

Remark: The result ensures that all the isotropic G-orbits, in particular also the Lagrangian orbits are contained in the level sets of the moment map. Since special Lagrangian submanifolds are in particular Lagrangian, we have the following corollary:

Corollary 4.3 *If L is a connected special Lagrangian submanifold in T^*S^n with symmetry group $G \subseteq SO(n+1)$, then $L \subseteq \mu^{-1}(c)$ for some $c \in Z(\mathfrak{g}^*)$, where $\mu : T^*S^n \to \mathfrak{g}^*$ is the moment map of the action of G.*

5 Homogeneous Special Lagrangian Submanifolds of T^*S^3

In this section we will show that the only homogeneous special Lagrangian 3-fold in T^*S^3 is the zero section $S^3 = \{(x, 0) \in \mathbb{R}^4 \times \mathbb{R}^4 | |x| = 1\}$.

We start by looking at the subgroups of $SO(4)$ that act on the complex quadric T^*S^3 with generic orbits of dimension 3. Let U be the group of unit quaternions and Φ the well-known $2:1$ homomorphism $\Phi \colon U \times U \to SO(4)$ given by $\Phi(u_1, u_2)(x) = u_1 x \bar{u}_2$. It is easy to see that $\mathfrak{so}(4) \cong \mathfrak{so}(3)_1 \oplus \mathfrak{so}(3)_2$, where $\mathfrak{so}(3)_1$ and $\mathfrak{so}(3)_2$ are two different copies of $\mathfrak{so}(3)$ whose intersection is the zero vector. By looking at the subgroups of $SO(4)$ of dimension ≥ 3 (see also [Io]), one can see that the only connected subgroups of $SO(4)$ that act on Q with generic orbits of dimension 3 are:

1. The full group SO(4) with Lie algebra $\mathfrak{so}(4)$ and whose infinitesimal generators are given by:

$$\left\{ \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\} \tag{5.19}$$

The generic orbit of the action on Q is an S^3.

2. The subgroup $\widetilde{SO(3)}$, with Lie algebra $\mathfrak{so}(3)_1$ (or $\mathfrak{so}(3)_2$), whose infinitesimal generators are given by:

$$\left\{ \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\} \tag{5.20}$$

3. The subgroup $S^1 \times SO(3)$, with Lie algebra $\mathfrak{so}(2)_1 \oplus \mathfrak{so}(3)_2$ (or $\mathfrak{so}(3)_1 \oplus \mathfrak{so}(2)_1$), whose infinitesimal generators are given by:

$$\left\{ \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\} \tag{5.21}$$

We prove the following result:

Proposition 5.1 *Every homogeneous special Lagrangian 3-fold in T^*S^3 is conjugate under the action of $SO(4)$ to the zero section $S^3 \subset T^*S^3$.*

Proof: Let L be a homogeneous special Lagrangian submanifold and $G \subset SO(4)$ its symmetry group. Then G is one of the three subgroups of $SO(4)$ described above. We consider each of the cases.

1. $G = SO(4)$. From equation (4.17), the moment map of the $SO(4)$-action is given by $\mu : Q \to \mathfrak{so}(4)^*$ with:

$$\mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_0\bar{z}_1), \mathrm{Im}(z_1\bar{z}_2), \mathrm{Im}(z_2\bar{z}_3), \mathrm{Im}(z_3\bar{z}_0), \mathrm{Im}(z_1\bar{z}_3), \mathrm{Im}(z_2\bar{z}_0))$$

Since $Z(\mathfrak{so}(4)^*) = \{0\}$, it follows from Corollary 4.3 that any $SO(4)$-invariant special Lagrangian 3-fold in Q^3 lies in the level set $\mu^{-1}(0)$. By applying an appropriate rotation with an element of $SO(4)$ one can assume that $x = (x_0, x_1, x_2, x_3) = (\cot t, \sin t, 0, 0),\ t \in [0, \pi)$. Now, since the special Lagrangian has to be in the zero level set of the moment map above, we have

$$\mathrm{Im}(z_0\bar{z}_1) = \mathrm{Im}(z_1\bar{z}_2) = \mathrm{Im}(z_2\bar{z}_3) = \mathrm{Im}(z_3\bar{z}_0) = \mathrm{Im}(z_1\bar{z}_3) = \mathrm{Im}(z_2\bar{z}_0)$$

Using the diffeomorphism (3.6) between the complex quadric Q^3 and T^*S^3, we see that ξ has to be of the form $\xi = \rho(-\sin t, \cos t, 0, 0)$, where $\rho = |\xi|$ and it has to also satisfy $\mathrm{Im}(z_0\bar{z}_1) = 0$, i.e. $\rho = 0$. Therefore, L is the zero section of the cotangent bundle.

2. $G = \widetilde{SO(3)}$. Note that the infinitesimal generator of G are some linear combinations of the infinitesimal generator of $SO(4)$. From equation (4.17), the moment map of the $\widetilde{SO(3)}$-action is given by $\mu : Q \to \mathfrak{so}(3)^*$ with:

$$\mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_0\bar{z}_1 + z_2\bar{z}_3), \mathrm{Im}(z_0\bar{z}_2 + z_3\bar{z}_1), \mathrm{Im}(z_0\bar{z}_3 + z_1\bar{z}_2))$$

Since $Z(\mathfrak{so}(3)^*) = \{0\}$, it follows from Corollary 4.3 that any $\widetilde{SO(3)}$-invariant special Lagrangian 3-fold in Q^3 lies in the level set $\mu^{-1}(0)$. Since the action of $\widetilde{SO(3)}$ which is given by left multiplication by unit quaternions is transitive on the unit sphere S^3, one can assume that $x = (x_0, x_1, x_2, x_3) = (\cot t, \sin t, 0, 0),\ t \in [0, \pi)$. Now, since the special Lagrangian has to be in the zero level set of the moment map above,we have

$$\mathrm{Im}(z_0\bar{z}_1 + z_2\bar{z}_3) = \mathrm{Im}(z_0\bar{z}_2 + z_3\bar{z}_1) = \mathrm{Im}(z_0\bar{z}_3 + z_1\bar{z}_2)$$

Using this and the diffeomorphism (3.6) yields:

$$\begin{aligned} x_0\xi_1 &= x_1\xi_0 \\ x_0\xi_2 &= x_1\xi_3 \\ x_0\xi_3 &= -x_1\xi_2 \end{aligned} \tag{5.22}$$

The last two equations yield: $\xi_2 = \xi_3 = 0$. Therefore, $\xi = \rho(-\sin t, \cos t, 0, 0)$, where $\rho = |\xi|$. First equation in (5.22) now implies: $\rho\cos^2 t = -\rho\sin^2 t$, i.e. $\rho = 0$.

3. $G = S^1 \times SO(3)$. An argument similar to the previous cases will also work and this is left to the reader to check. $\square$

6 Cohomogeneity one Special Lagrangian Submanifolds in T^*S^3

The next most symmetric case is when the symmetry group of the special Lagrangian submanifold acts with cohomogeneity one. In this case, the differential equation of a special Lagrangian simplifies and we can hope to find examples by solving an O.D.E. The idea is to find subgroups G of $SO(4)$ whose generic orbits in T^*S^3 are of dimension 2. In order for the generic orbit to be dimension 2, one must have that $\dim G \geq 2$. Then, the strategy is to find an extra direction in which the submanifold will become special Lagrangian. We will do this in this section.

The subgroups of $SO(4)$ that act on Q with orbits of dimension 2 are the subgroup $SO(3)$ which leaves a direction invariant and the maximal torus T^2. We start out with the maximal torus case, which is the most interesting case and our main result.

6.1 The T^2-case

Let G be the maximal torus T^2 of $SO(4)$, described as:

$$\left\{ \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix},\ \theta_1, \theta_2 \in [0, 2\pi) \right\}$$

The following result gives all the special Lagrangian 3-folds of T^*S^3 invariant under the action of T^2.

Theorem 6.1 *The special Lagrangian submanifolds in $T^*S^3 = Q = \{z \in \mathbb{C}^4 \big| \sum_{i=0}^3 z_i^2 = 1\}$ with the Calabi-Yau metric, which are invariant under the action of the maximal torus T^2 of $SO(4)$ are given by the equations:*

$$\begin{aligned} u'(|z|^2)\ \mathrm{Im}(z_0\bar{z_1}) &= c_1 \\ u'(|z|^2)\ \mathrm{Im}(z_2\bar{z_3}) &= c_2 \\ \mathrm{Im}(z_0^2 + z_1^2) &= c_3 \end{aligned} \tag{6.23}$$

where u is given by (3.9) and c_1, c_2 and c_3 are any real constants.

Proof:
Using the two infinitesimal generators of the T^2-action:

$$B_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and the expression (4.17), the moment map for the T^2-action on Q^3 is given by:

$$\mu : Q^3 \to (\mathfrak{t}^2)^*,\ \ \mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_0\bar{z_1}), \mathrm{Im}(z_2\bar{z_3}))$$

Since $(\mathfrak{t}^2)^* = \mathbb{R}^2$, it follows from Corollary 4.3 that any T^2-invariant special Lagrangian 3-fold L in Q^3 lies in a level set $\mu^{-1}(c)$, where $c = (c_1, c_2) \in \mathbb{R}^2$. The first two equations enforce this condition and ensures that the submanifold is Lagrangian. One can also check directly that $\omega_{St}\mid_L = 0$ using expression (3.11). In order to impose the special Lagrangian condition at a given point z, we compute Ω_{St} on the three tangent vectors $Y_1 = B_1\, z, Y_2 = B_2\, z$ and $Y_3 = \dot{z}$:

$$\Omega_{St}(Y_1, Y_2, Y_3) = (dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(z, Y_1, Y_2, Y_3) = \begin{vmatrix} z_0 & -z_1 & 0 & \dot{z_0} \\ z_1 & z_0 & 0 & \dot{z_1} \\ z_2 & 0 & -z_3 & \dot{z_2} \\ z_3 & 0 & z_2 & \dot{z_3} \end{vmatrix}$$
$$= (z_0^2 + z_1^2)(z_2\dot{z_2} + z_3\dot{z_3}) - (z_2^2 + z_3^2)(z_0\dot{z_0} + z_1\dot{z_1})$$

Now using $z_0^2+z_1^2+z_2^2+z_3^2 = 1,\ \mathrm{Im}(z_0^2+z_1^2) = -\mathrm{Im}(z_2^2+z_3^2)$ and $\mathrm{Im}(z_2\dot{z_2}+z_3\dot{z_3}) = -\mathrm{Im}(z_0\dot{z_0}+z_1\dot{z_1})$ we finally obtain $\mathrm{Im}\,\Omega_{St}(Y_1, Y_2, Y_3) = \mathrm{Im}(z_0\dot{z_0} + z_1\dot{z_1})$, from which the last equation follows. □

Remark: Equations (6.23) are obviously T^2-invariant and linearly independent. Therefore, the above family of T^2-invariant special Lagrangian 3-folds foliate the cotangent bundle of the sphere, including the zero section. The generic orbit is $T^2 \times \mathbb{R}$ where T^2 is an orbit of the maximal torus in the isometry group.

One can also view the special Lagrangian 3-folds above as being obtained by rotating a curve in $T^*S^3 = \{(x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^4,\ |x| = 1,\ x \cdot \xi = 0\}$ by the torus action. To see this, let $\gamma(t) = (x(t), \xi(t)) \in T^*S^3$ be a curve in the complex quadric. By applying an appropriate rotation with an element of T^2, we can assume that $x(t) = \begin{pmatrix} \cos t \\ 0 \\ \sin t \\ 0 \end{pmatrix}, t \in [0, \pi)$ (since $|x| = 1$). Denote the length of the vector $\xi = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ by $\rho = |\xi| \geq 0$. Let $\rho_0 = \xi_0^2 + \xi_2^2$ and $\rho_1 = \xi_1^2 + \xi_3^2$. Since $\rho_0^2 + \rho_1^2 = \rho^2$, we let:

$$\rho_0 = \rho\cos\varphi$$
$$\rho_1 = \rho\sin\varphi$$

Since $x \cdot \xi = 0$, we can parameterize the vector as $\xi(t) = \begin{pmatrix} -\rho_0 \sin t \\ \rho_1 \cos\psi \\ \rho_0 \cos t \\ \rho_1 \sin\psi \end{pmatrix} = \rho \begin{pmatrix} -\cos\varphi\sin t \\ \sin\varphi\cos\psi \\ \cos\varphi\cos t \\ \sin\varphi\sin\psi \end{pmatrix}$. Using the diffeomorphism h given by relation (3.6), one gets that any point on the quadric is conjugate under the T^2-action to a point of the form:

$$z = \begin{pmatrix} \cos t\cosh\rho - i\sinh\rho\cos\varphi\sin t \\ i\sinh\rho\sin\varphi\cos\psi \\ \sin t\cosh\rho + i\sinh\rho\cos\varphi\cos t \\ i\sinh\rho\sin\varphi\sin\psi \end{pmatrix} \in Q$$

Note that $|z|^2 = \cosh(2\rho)$.

In fact the whole quadric Q^6 can be parametrized as:

$$\begin{pmatrix} \cos\theta_1 \cos t \cosh\rho - i\,(\cos\theta_1 \sinh\rho \cos\varphi \sin t + \sin\theta_1 \sinh\rho \sin\varphi \cos\psi) \\ \sin\theta_1 \cos t \cosh\rho + i\,(\cos\theta_1 \sinh\rho \sin\varphi \cos\psi - \sin\theta_1 \sinh\rho \cos\varphi \sin t) \\ \cos\theta_2 \sin t \cosh\rho + i\,(\cos\theta_2 \sinh\rho \cos\varphi \cos t - \sin\theta_2 \sinh\rho \sin\varphi \sin\psi) \\ \sin\theta_2 \sin t \cosh\rho + i\,(\sin\theta_2 \sinh\rho \cos\varphi \cos t + \cos\theta_2 \sinh\rho \sin\varphi \sin\psi) \end{pmatrix}$$

where $t, \theta_1, \theta_2, \varphi, \psi \in S^1$ and $\rho \geq 0$. Equations (6.23) become:

$$\begin{aligned} &u'(\cosh(2\rho)) \sinh(2\rho) \cos t \sin\varphi \cos\psi = c_1 \\ &u'(\cosh(2\rho)) \sinh(2\rho) \sin t \sin\varphi \sin\psi = c_2 \\ &\sinh(2\rho) \sin(2t) \cos\varphi = c_3 \end{aligned} \tag{6.24}$$

These equations are independent of the torus parameters θ_1, θ_2 and describe a curve in the parameter space (ρ, t, φ, ψ) which under the T^2-action on Q gives a family of special Lagrangian 3-folds L_c in the complex quadric, where $c = (c_1, c_2, c_3)$. These special Lagrangians have the topology of $T^2 \times \mathbb{R}$ and foliate the ambient space Q.

Remark: Another method to find the T^2-invariant special Lagrangians is to start with a curve in the (ρ, t, φ, ψ)-space which lies in the level set of the moment map and write an O.D.E. to impose the special Lagrangian condition. This O.D.E. is in general difficult to solve explicitly and the implicit method of proving theorem 6.1 is much simpler.

Asymptotic Behaviour

We now study the asymptotic behaviour of this family of special Lagrangian, i.e. the limiting behavior of the family as $\rho = |\xi| \to \infty$. As we have seen previously, the cotangent bundle $T^*S^3 = \{(x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^4,\ |x| = 1,\ x \cdot \xi = 0\}$ approaches the conifold Q_0 asymptotically as $|\xi| \to \infty$.

Notice that as $\rho \to \infty$, $u'(\cosh(2\rho)) \to \infty$ from relation (3.8), so in the limit, equations (6.24) become one of the following cases:

$$\begin{aligned} &a) \sin\varphi = 0,\ \sin t = 0 \\ &b) \sin\varphi = 0,\ \cos t = 0 \\ &c) \cos t = 0,\ \sin\psi = 0 \\ &d) \cos\psi = 0,\ \sin t = 0 \end{aligned} \tag{6.25}$$

We will study each case separately.

a) $\sin\varphi = 0,\ \sin t = 0 \Rightarrow z = \begin{pmatrix} \cosh\rho \\ 0 \\ i\sinh\rho \\ 0 \end{pmatrix}$. The unit vector $\frac{z}{|z|}$ is:

$$\frac{z}{|z|} = \frac{1}{\sqrt{\cosh(2\rho)}} \begin{pmatrix} \cosh\rho \\ 0 \\ i\,\sinh\rho \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ i \\ 0 \end{pmatrix} \in Q_0, \quad \text{as } \rho \to \infty$$

Applying the T^2-action in the limit, one gets a surface Σ diffeomorphic to T^2 and Σ is a submanifold of the conifold Q_0:

$$\Sigma = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ i\,\cos\theta_2 \\ i\,\sin\theta_2 \end{pmatrix}, \theta_1, \theta_2 \in [0, 2\pi) \right\} \subset Q_0$$

We will show that the cone on Σ, $C(\Sigma) = \{sz | z \in \Sigma,\ s \in \mathbb{R}\}$, is special Lagrangian in the conifold Q_0, endowed with the Ricci-flat metric found by Candelas and de la Ossa[CO].
We first show that the cone $C(\Sigma)$ is Lagrangian, i.e. $\omega_{cone}\,|_{C(\Sigma)} = 0$. The moment map of the T^2-action on the cone is:

$$\mu_0 : Q_0 \to \mathbb{R}^2,\ \ \mu_0(z) = u'_{cone}(|z|^2)(\mathrm{Im}(z_0\bar{z}_1), \mathrm{Im}(z_2\bar{z}_3))$$

where $u_{cone}(r^2)$ is the potential function for the conifold given in section 3.2. Since the cone on Σ is seen to lie in $\mu_0^{-1}(0,0)$, it follows from Prop. 4.2 that it is Lagrangian.

Next we show that the cone is special Lagrangian, i.e. $\mathrm{Im}\,\Omega_{cone}\,|_{C(\Sigma)} \equiv 0$. For this we compute Ω_{cone} on three tangent vectors Y_1, Y_2, Y_3 to the cone $C(\Sigma)$. One of them is the position vector and the other two vectors are the derivatives with respect to the parameters θ_1 and θ_2. The unit vector normal to the cone is given by

$$w = \bar{z} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \\ -i\,\cos\theta_2 \\ -i\,\sin\theta_2 \end{pmatrix}$$

and this is the vector we will use to compute Ω_{cone} as follows:

$$\begin{aligned} \mathrm{Im}\,\Omega_{cone}(Y_1, Y_2, Y_3) &= \mathrm{Im}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(\bar{z}, Y_1, Y_2, Y_3) \\ &= \mathrm{Im}\,\frac{1}{2s^2} \begin{vmatrix} \frac{\cos\theta_1}{\sqrt{2}} & \frac{\cos\theta_1}{\sqrt{2}} & -s\frac{\sin\theta_1}{\sqrt{2}} & 0 \\ \frac{\sin\theta_1}{\sqrt{2}} & \frac{\sin\theta_1}{\sqrt{2}} & s\frac{\cos\theta_1}{\sqrt{2}} & 0 \\ -\frac{i\cos\theta_2}{\sqrt{2}} & \frac{i\cos\theta_2}{\sqrt{2}} & 0 & -s\frac{i\sin\theta_2}{\sqrt{2}} \\ -\frac{i\sin\theta_2}{\sqrt{2}} & \frac{i\sin\theta_2}{\sqrt{2}} & 0 & s\frac{i\cos\theta_2}{\sqrt{2}} \end{vmatrix} = 0 \end{aligned}$$

Since $\mathrm{Im}\,\Omega_{cone}\,|_{C(\Sigma)} \equiv 0$, the cone on Σ is special Lagrangian.

Remark: The above analysis shows that in this case, the T^2-invariant special Lagrangian in the quadric goes asymptotically to a special Lagrangian cone in the conifold.

b) $\sin\varphi = 0,\ \cos t = 0$. In this case, in the limit as $\rho \to \infty$, the unit vector $\frac{z}{|z|}$ goes to:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix} \in Q_0$$

and the conclusion is the same as in case a). The unit vector $\frac{z}{|z|}$ in this case is the unit vector from case a) rotated by J and so the the limit is the same special Lagrangian cone.

c) $\cos t = 0,\ \sin\psi = 0 \Rightarrow z = \begin{pmatrix} -i\sinh\rho\cos\varphi \\ i\sin\varphi\sinh\rho \\ \cosh\rho \\ 0 \end{pmatrix}$. The unit vector $\frac{z}{|z|}$ goes in the limit to:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} -i\cos\varphi \\ i\sin\varphi \\ 1 \\ 0 \end{pmatrix} \in Q_0$$

Applying the T^2-action in the limit, one gets a surface Σ diffeomorphic to T^2 and Σ is a submanifold of the conifold Q_0:

$$\Sigma = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} i\,\cos\theta_1 \\ i\,\sin\theta_1 \\ \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}, \theta_1, \theta_2 \in [0, 2\pi) \right\} \subset Q_0$$

Same argument as in case a) shows that the cone on Σ is special Lagrangian in the conifold Q_0.

d) $\cos\psi = 0,\ \sin t = 0$. This case yields the same special Lagrangian cone as in case c) and it is left to the reader.

Remark 1: The special Lagrangian family in the complex quadric is asymptotic to a special Lagrangian cone on flat tori in the conifold. For a fixed $c = (c_1, c_2, c_3)$, L_c has two components, each of them asymptotic to the special Lagrangian cone in the conifold.

Remark 2: When $\rho = 0$, equations (6.24) are identically satisfied and the solution is the zero section S^3 of the cotangent bundle, which is well-known to be special Lagrangian.

Remark 3: If we set $c_1 = c_2 = 0$ in equation (6.24), we obtain the equation

$$\sin(2t)\sinh(2\rho) = c$$

in the (t, ρ)-plane, $t \in [0, \pi)$. The special Lagrangian in this case lies in the zero-level set of the moment map. The phase portrait of the equation in this special case is shown in Figure 1. We can see that as $t \to \frac{\pi}{2}$, $|\rho| \to \infty$. The special Lagrangians are obtained by rotating these curves by a T^2-action.

6.2 The $SO(3)$-case

Let G be the subgroup $SO(3, \mathbb{R})$ of $SO(4, \mathbb{C})$, where $SO(3, \mathbb{R})$ sits in $SO(4, \mathbb{R})$ as matrices of the form

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix},\ A \in SO(3) \right\}$$

and $SO(4, \mathbb{R})$ is embedded in $SO(4, \mathbb{C})$ as 4×4 real matrices. Note that this $SO(3)$ is embedded differently in $SO(4)$ than the subgroup $\widetilde{SO(3)}$ from section 5.

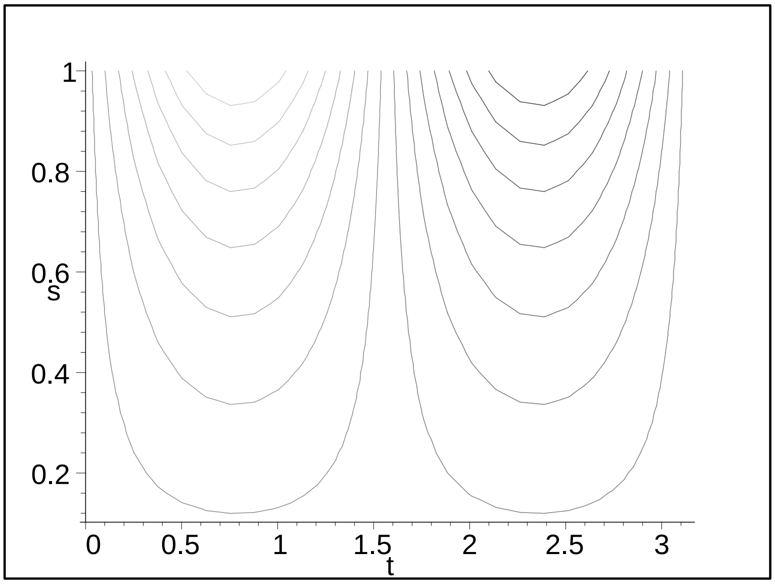

Figure 1: Phase Portrait for the equation $\sin(2t)\sinh(2s) = c$

Theorem 6.2 *Let* $T^*S^3 = \{(x, \xi) \in \mathbb{R}^4 \times \mathbb{R}^4,\ |x| = 1,\ x \cdot \xi = 0\}$*. Then,*

$$L_c = \{(gx, g\xi) \mid (x, \xi) \in T^*S^3,\ g \in SO(3),\ x = (\cos t, \sin t, 0, 0),\ 2|\xi| - \cos(2t)\sinh(2|\xi|) = c\}$$

is the only $SO(3)$*-invariant special Lagrangian* 3*-fold of* T^*S^3*.*

Proof: Let $z = (z_0, z_1, z_2, z_3) \in Q^3$, i.e. $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$. Using expression (4.17), the moment map of the $SO(3)$-action on Q^3 is given by:

$$\mu : Q^3 \to \mathfrak{so}(3)^*,\ \mu(z_0, z_1, z_2, z_3) = u'(|z|^2)(\mathrm{Im}(z_1\bar{z_2}), \mathrm{Im}(z_2\bar{z_3}), \mathrm{Im}(z_3\bar{z_1}))$$

To see how we obtain this, let

$$A_1 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix}$$

be the infinitesimal generators of the $SO(3)$-action. Equation (4.17) implies that:

$$\mu_{A_3}(z) = \frac{1}{2}u'(|z|^2) < A_3 z, iz >= \frac{1}{2}u'(|z|^2)\left\langle \begin{pmatrix} 0\\ -z_2\\ z_1\\ 0 \end{pmatrix}, \begin{pmatrix} iz_0\\ iz_1\\ iz_2\\ iz_3 \end{pmatrix} \right\rangle =$$

$$= \frac{1}{2}u'(|z|^2)(i\bar{z_1}z_2 - iz_1\bar{z_2}) = u'(|z|^2)\,\mathrm{Im}(z_1\bar{z_2})$$

Similarly, $\mu_{A_2}(z) = u'(|z|^2)\,\mathrm{Im}(z_3\bar{z_1})$, $\mu_{A_1}(z) = u'(|z|^2)\,\mathrm{Im}(z_2\bar{z_3})$ and we get the moment map of the $SO(3)$-action on Q^3.

Since $Z(\mathfrak{so}(3)^*) = \{0\}$, it follows from Corollary 4.3 that any $SO(3)$-invariant special Lagrangian 3-fold in Q^3 lies in the level set $\mu^{-1}(0)$. By applying an appropriate rotation with an element of $SO(3)$, we can assume that $x = (x_0, x_1, x_2, x_3) = (\cot t, \sin t, 0, 0),\ t \in [0, \pi)$. Now, since the special Lagrangian has to lie in the zero level set of the moment map,we have $\mathrm{Im}(z_1\bar{z}_2) = \mathrm{Im}(z_2\bar{z}_3) = \mathrm{Im}(z_3\bar{z}_1) = 0$ and hence the level set $\mu^{-1}(0)$ is given by:

$$\mu^{-1}(0) = \{\, (gx, g\xi) | g \in SO(3),\ x(t) = (\cos t, \sin t, 0, 0),\ \xi(t) = \rho(-\sin t, \cos t, 0, 0),\ t \in [0, \pi),\ \rho \geq 0\,\}$$

or equivalently, using the identification of T^*S^3 with Q,

$$\left\{ g.z(t,\rho)\ :\ g \in SO(3),\ z(t,\rho) = \begin{pmatrix} \cos t\cosh\rho - i\sin t\sinh\rho \\ \sin t\cosh\rho + i\cos t\sinh\rho \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\tau) \\ \sin(\tau) \\ 0 \\ 0 \end{pmatrix},\ \rho \geq 0,\ t \in [0,\pi) \right\}$$

where $\tau = t + i\rho$ lies in the $[0, \pi)$ vertical strip of the complex plane.

We now look for curves $\gamma(s)$ in the $\tau = (t, \rho)$ plane which after applying the $SO(3)$-action give rise to special Lagrangians of the form $L = g.\gamma(s)$. If $p = \gamma(s) = \begin{pmatrix} \cos(\tau(s)) \\ \sin(\tau(s)) \\ 0 \\ 0 \end{pmatrix}$ is a point on the level set $\mu^{-1}(0)$, then $A_3\, p = \begin{pmatrix} 0 \\ 0 \\ \sin(\tau) \\ 0 \end{pmatrix}$, $A_2\, p = \begin{pmatrix} 0 \\ \\ 0 \\ -\sin(\tau) \end{pmatrix}$ and $A_1\, p = 0$, where A_1, A_2, A_3 are the infinitesimal generators of $\mathfrak{so}(3)$ as defined above.

The tangent plane at p to L is spanned by the vectors $< X_1 = A_1\, p, X_2 = A_2\, p, X = \dot{\gamma}(s) >$ at p, where $\dot{\gamma}(s) = \begin{pmatrix} -\sin(\tau)\,\dot{\tau} \\ \cos(\tau)\,\dot{\tau} \\ 0 \\ 0 \end{pmatrix}$.

L is invariant under the $SO(3)$-flow and $\omega_{St}|_L = 0$, since it lies in the zero level set of the moment map. Therefore L is Lagrangian. One can also verify this directly with formula (3.11). Now we will impose the condition that L is special Lagrangian, i.e. $\mathrm{Im}\,\Omega_{St} = 0$ should hold.

Using equation (3.12), we compute $\Omega_{St}(X_1, X_2, X_3) = \big(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3\big)\big(\gamma(s), X_1, X_2, \dot{\gamma}(s)\big)$:

$$\begin{vmatrix} \cos(\tau) & 0 & 0 & -\sin(\tau)\,\dot{\tau} \\ \sin(\tau) & 0 & 0 & \cos(\tau)\,\dot{\tau} \\ 0 & \sin(\tau) & 0 & 0 \\ 0 & 0 & -\sin(\tau) & 0 \end{vmatrix} = -\sin^2(\tau)\,\dot{\tau}$$

Integrating, the condition $\mathrm{Im}\,\Omega_{St} = 0$ becomes:

$$\mathrm{Im}(2\tau - \sin(2\tau)) = c$$

which is equivalent to

$$2\rho - \cos(2t)\sinh(2\rho) = c \tag{6.26}$$

where c is any real constant. $\square$

Remark 1 : Notice that $\rho = 0$ is a solution to equation (6.26) and the special Lagrangian obtained is the zero section of T^*S^3, which is known to be special Lagrangian.

Remark 2: The result we obtained in the $n = 3$ case can be generalized to study the $SO(n)$-invariant special Lagrangian n-folds in T^*S^n. We will consider this case in the next section.

Remark 3: This family of $SO(n)$-invariant special Lagrangian submanifolds was also obtained by Anciaux [An] using different methods.

The $SO(3)$-invariant special Lagrangian L can also be written intrinsically as follows. Choose (z_0, z_1, z_2, z_3) coordinates on the complex quadric Q^3. Then L is given by the equations:

$$\begin{aligned} u'(|z|^2)\,\mathrm{Im}(z_1\bar{z}_2) &= c_1 \\ u'(|z|^2)\,\mathrm{Im}(z_2\bar{z}_3) &= c_2 \\ \mathrm{Im}\left(\arccos(z_0) - z_0\sqrt{1-z_0^2}\right) &= c \end{aligned} \tag{6.27}$$

where c_1, c_2, c are constants.

Asymptotic Behaviour:

In what follows, we will study the equation $2\rho - \cos(2t)\sinh(2\rho) = c$ in the (t,ρ)-plane and describe the asymptotic behaviour of the special Lagrangian 3-folds obtained in theorem 6.2.

When $\rho \to \infty$, equation (6.26) becomes $\cos 2t = 0$, so $t = \frac{\pi}{4}$. The unit vector $\frac{z}{|z|}$ is:

$$\frac{1}{\sqrt{2\cosh(2\rho)}}\begin{pmatrix}\cosh\rho - i\,\sinh\rho \\ \cosh\rho + i\,\sinh\rho \\ 0 \\ 0\end{pmatrix} \to \frac{1}{2}\begin{pmatrix}1-i \\ 1+i \\ 0 \\ 0\end{pmatrix} \in Q_0, \quad \text{as } \rho \to \infty$$

Applying the $SO(3)$-action in the limit, one gets a surface Σ diffeomorphic to S^2 and Σ is a submanifold of the conifold Q_0. As before, the cone on Σ is special Lagrangian in the conifold Q_0, endowed with the Ricci-flat metric found by Candelas and de la Ossa[CO]. To see this, we will first show that the cone $C(\Sigma)$ is Lagrangian, i.e. $\omega_{cone}\,|_{C(\Sigma)} = 0$. The moment map of the $SO(3)$-action on the cone is:

$$\mu_0 : Q_0 \to \mathbb{R}^3,\ \mu_0(z) = u'_{cone}(|z|^2)(\mathrm{Im}(z_1\bar{z}_2), \mathrm{Im}(z_2\bar{z}_3), \mathrm{Im}(z_3\bar{z}_1))$$

where $u_{cone}(r^2)$ is the potential function for the conifold given in section 3.2. Since the cone on Σ is seen to lie in $\mu_0^{-1}(0,0)$, it follows from Prop. 4.2 that it is Lagrangian.

Next we show that the cone is special Lagrangian, i.e. $\mathrm{Im}\,\Omega_{cone}\,|_{C(\Sigma)} \equiv 0$. For this we compute Ω_{cone} on three tangent vectors Y_1, Y_2, Y_3 to the cone $C(\Sigma)$. One of them is the

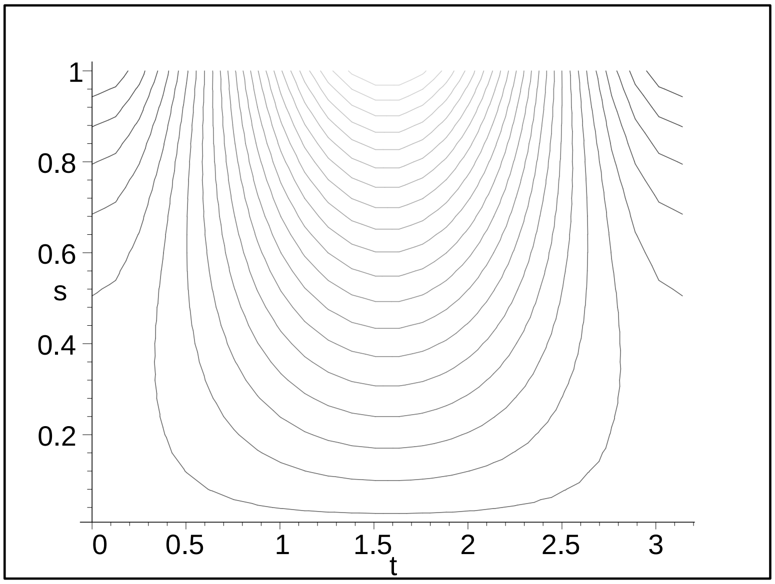

Figure 2: Phase Portrait for the equation $2s - \cos(2t)\sinh(2s) = c$

position vector and the other two vectors are the vectors $A_3 z$ and $A_2 z$, where A_2, A_3 are the infinitesimal generators as defined above. The unit vector normal to the cone is given by

$$w = \bar{z} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \\ 0 \\ 0 \end{pmatrix}$$

and this is the vector we will use to compute Ω_{cone} as follows:

$$\begin{aligned} \operatorname{Im}\Omega_{cone}(Y_1, Y_2, Y_3) &= \operatorname{Im}(dz_0 \wedge dz_1 \wedge dz_2 \wedge dz_3)(\bar{z}, Y_1, Y_2, Y_3) \\ &= \operatorname{Im}\frac{1}{8}\begin{vmatrix} 1-i & 0 & 0 & 1+i \\ 1+i & 0 & 0 & 1-i \\ 0 & 1+i & 0 & 0 \\ 0 & 0 & -1-i & 0 \end{vmatrix} = 0 \end{aligned}$$

Since $\operatorname{Im}\Omega_{cone}\mid_{C(\Sigma)} \equiv 0$, the cone on Σ is special Lagrangian.

Remark: The above analysis shows that in this case, the $SO(3)$-invariant special Lagrangians in the quadric approach asymptotically a special Lagrangian cone on S^2 in the conifold. This limiting S^2 can be explicitly described in coordinates as follows:

$$z = \frac{1}{2}\begin{pmatrix} 1-i \\ (1+i)\cos\varphi\cos\theta \\ (1+i)\cos\varphi\sin\theta \\ (1+i)\sin\varphi \end{pmatrix} \qquad \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}],\ \theta \in [0, 2\pi)$$

Remark: In particular, in the case of Eguchi-Hanson metric on Q^2 ($n = 2$), the subgroups $SO(2)$ and T^1 of $SO(3)$ coincide and equations (6.23) and (6.27) give the same special Lagrangian submanifold invariant under $SO(2)$, embedded in $SO(3)$ as:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}, \ A \in SO(2) \right\}$$

In coordinates $z = (z_0, z_1, z_2)$ on Q, these special Lagrangians are given by the equations:

$$\begin{aligned} &\frac{|z|}{\sqrt{|z|^2+1}} \operatorname{Im}(z_1 \bar{z}_2) = c_1 \\ &\operatorname{Im}(z_0) = c_2 \end{aligned}$$

where c_1, c_2 are constants.

6.3 The general $SO(n)$-case

One can generalize our method to higher dimensions and recover the $SO(n)$-invariant family of special Lagrangians that H. Anciaux obtained in [An]. Computing the imaginary part of the holomorphic n-form Ω_{St} to check the special Lagrangian condition yields

$$\operatorname{Im}(\sin^{n-1}(\tau)\dot{\tau}) = 0$$

Now let $F(\tau)$ be the function $\operatorname{Im}\left(\int_0^\tau sin^{n-1}(\sigma)d\sigma\right)$. Combining with the moment map conditions, the $SO(n)$-invariant special Lagrangians are given by the following set of equations:

$$\begin{aligned} &u'(|z|^2) \operatorname{Im}(z_1 \bar{z}_j) = c_j \qquad 2 \le j \le n \\ &\operatorname{Im}\left(F(\arccos(z_0))\right) = c \end{aligned} \tag{6.28}$$

where and c_j, c are constants and the function u satisfies (3.7) for the given dimension.

7 Concluding remarks:

We conclude the paper with a remark and some comments.

Remark: There are no special Lagrangian submanifolds in T^*S^n which are graphs over the zero section S^n.

Proof: Let $L = \left\{(x, \xi(x)) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} | \ |x| = 1, \ x \cdot \xi = 0\right\}$ be a special Lagrangian such that L is a graph over the zero section S^3. Then L is minimal and is homotopic to the zero section. Since a special Lagrangian is absolutely area minimizing in its homology class, it follows that L is the zero section S^n.

The SYZ conjecture [SYZ] explains mirror symmetry by analyzing "dual" fibrations of two different Calabi-Yau 3-folds by special Lagrangian tori T^3. Since in our case we are dealing with a non-compact Calabi-Yau, we obtain a $T^2 \times \mathbb{R}$-fibration of the deformed conifold instead of a T^3. It is known that the local mirror of T^*S^3 is the $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$-bundle over $\mathbb{C}P^1$. In our next paper, we plan to investigate what happens to these special Lagrangians under the conifold transition to the mirror.

References

[An] H. Anciaux, *Special Lagrangian submanifolds in the complex sphere*, arXiv:math.DG/0311288

[BS] R. L. Bryant, S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. 58 (1989), no. 3, 829-850

[CO] P. Candelas, X. de la Ossa *Comments on conifolds*, Nuclear Physics B342 (1990), 246-268

[DS] A. Dancer, I.A.B. Strachan, *Eistein Metrics on tangent bundles of sphere*, Classical Quantum Gravity 19 (2002), no.18, 4663-4670

[EH] T. Eguchi, A.J. Hanson, *Asymptotically flat self-dual solutions to Euclidean gravity*, Phys. Lett B74 (1978), 249-251

[Ha] M. Haskins, *Special Lagrangian cones*, Amer. J. Math. 126 (2004), no.4, 845-871

[J1] D. D. Joyce, Compact Manifolds with Special Holonomy, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000

[J2] D. D. Joyce, *Special Lagrangian m-folds in* $\mathbb{C}^m$ *with symmetries* , Duke Math Journal **115** (2002), no.1, 1-51

[J3] D. D. Joyce, *Special Lagrangian 3-folds and integrable systems*, arXiv:math.DG/0101249.

[J4] D. D. Joyce, *Ruled special Lagrangian 3-folds in* $\mathbb{C}^3$, Proceedings of the London Mathematical Society 85 (2002), 233-256

[J5] D. D. Joyce, *Constructing special Lagrangian m-folds in* $\mathbb{C}^m$ *by evolving quadrics*, Math. Ann. **320** (2001), no.4, 757-797 pre01660683

[HL] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Mathematica 148 (1982),47-157

[Io] M. Ionel, *Second Order Families of Special Lagrangian Submanifolds in* $\mathbb{C}^4$, J. Differential Geometry 65 (2003), 211-272

[IKM] M. Ionel, S. Karigiannis and M. Min-Oo, *Bundle constructions of calibrated submanifolds in* $\mathbb{R}^7$ *and* $\mathbb{R}^8$, to appear in Mathematical Research Letters

[KM] S. Karigiannis and M. Min-Oo, *Calibrated subbundles in non-compact manifolds of special holonomy*, arXiv:math.DG/0412312.

[ML] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. 6 (1998), 705-747

[CGLP] M. Cvetic, G. W. Gibbons, H. Lu, C. N. Pope, *Special Holonomy Spaces and M-theory*, Unity from duality: gravity, gauge theory and strings (Les Houches, 2001), 523-545, NATO Adv. Study Inst., EDP Sci., Les Ulis, 2003

[SW] R. Schoen and J. G. Wolfson, *Minimizing area among Lagrangian surfaces: the mapping problem*, J. Differential Geom. **58** (2001), no.1, 1-86

[Sz] R. Szöke, *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. 291 (1991), 409-428

[St] M. Stenzel, *Ricci-flat metrics on the complexification of a compact rank one symmetric space*, Manuscripta Math. 80 (1993), no.2, 151-163

[SYZ] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Physics B479 (1996), no.1-2, 243-259

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