VAFA–WITTEN BOUND ON THE COMPLEX PROJECTIVE SPACE

HÉLÈNE DAVAUX & MAUNG MIN-OO

ABSTRACT. We prove that the largest first eigenvalue of the Dirac operator among all hermitian metrics on the complex projective space of odd dimension m, larger than the Fubini-Study metric is bounded by $(2m(m+1))^{1/2}$.

1. INTRODUCTION

Lower and upper eigenvalue estimates for the Dirac operator on a closed Riemannian spin manifold are derived by very different methods. Lower bound estimate are usually based on a Bochner-Lichnerowicz-Weitzenböck formula and related to scalar curvature ([Fri80, Kir90, Mor95, Sem99]). In the other direction, C. Vafa and E. Witten [VW84, Ati85] showed that there exists a common upper bound for the smallest eigenvalue of all *twisted* Dirac operators on a given Riemannian manifold. The common upper bound they found depends on the metric of the base manifold but is independent of the bundle and the unitary connection that is used to twist the Dirac operator.

The idea is as follows. Compare the Dirac operator $\mathcal{D}^0$ (or a multiple of it) to a twisted Dirac operator $\mathcal{D}^1$ acting on sections of the same vector bundles. By index theory make sure that $\mathcal{D}^1$ has a kernel. Let k the multiplicity of the eigenvalue 0 of $\mathcal{D}^1$. Estimate the norm of the difference (which is a zero order operator), $\|\mathcal{D}^0 - \mathcal{D}^1\| = \|L\|$, by geometric quantities. Then at least k eigenvalues of $\mathcal{D}^0$ are bounded by $\|L\|$.

Using this method, H. Baum [Bau91] exhibited later an explicit upper bound for the first eigenvalue of the (untwisted) Dirac operator on an even-dimensional Riemannian manifold that can be sent on a sphere by a map of (high) non-zero degree. This bound depends on the Lipschitz norm of the map from the manifold to the round sphere. For the tori, there is also the paper of N. Anghel [Ang00].

In [Her04], Marc Herzlich gives an optimal result for the first eigenvalue of the Dirac operator on the spheres. The goal of this short note is to find an analogous result for the complex projective spaces.

Theorem 1.1. *Let g be any hermitian metric and f be the Fubini-Study metric of maximal sectional curvature 4 on the complex projective space $\mathbb{C}P^m$, with m odd.*

If $g \geq f$, that is $g(v,v) \geq f(v,v)$ for all $v \in T\mathbb{C}P^m$, then

$$\lambda_1(g)^2 \leq 2m(m+1)$$

where $\lambda_1(g)$ is the lower eigenvalue of the Dirac operator of g.

There are also some techniques (the metric Lie derivative) described by J-P. Bourguignon and P. Gauduchon [BG92] to compare spinors for different metrics. In particular,

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they prove some results on eigenvalue of Dirac operator when the metric varies. In the case of conformal change of the metric, we can give the following corollary of O. Hijazi work [Hij86]:

Proposition 1.2. *Let (M, g_0) be a closed spin manifold. If $g = e^u g_0$ where u is a non negative function on M, then $|\lambda_1(g)| \leq |\lambda_1(g_0)|$.*

We still do not know whether Theorem 1.1 can be improved so that the result is optimal for the Fubini-Study metric. That is, can we prove that, for $g \geq f$, we have

$$\lambda_1(g)^2 \leq (m+1)^2 = \lambda_1^2(f)$$

with equality if and only if $f = g$.

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2. Background material

2.1. **Tautological bundle.** Let h be the standard hermitian product on $\mathbb{C}^{m+1}$. We denote $\langle\ ,\ \rangle = \Re e h$. It is the standard riemannian product on $\mathbb{R}^{2m+2} = \mathbb{C}^{m+1}$.

If $\mathbb{C}P^m = \{[z], z \in S^{2m+1} \subset \mathbb{C}^{m+1}\}$, then the tautological bundle is given by

$$E = \{([z], w), \quad w \in [z]\} \subset \mathbb{C}P^m \times \mathbb{C}^{m+1}.$$

We consider the complement F of E:

$$F = \{([z], w), \quad w \perp_h z\} \subset \mathbb{C}P^m \times \mathbb{C}^{m+1}.$$

We have $E \oplus F = \mathbb{C}P^m \times \mathbb{C}^{m+1}$. Coming from $\mathbb{C}^{m+1}$, there are an hermitian structure on the bundles E and F.

We denote by α the standard identification between $T\mathbb{C}P^m$ and $\mathrm{Hom}_{\mathbb{C}}(E, F)$ defined as follow: Let X be a vector of $T_x\mathbb{C}P^m$, X is generated by the curve $x(t) \in \mathbb{C}P^m$ such that $x(0) = x$ and $\dot{x}(0) = X$. We associate to $X \in T_x\mathbb{C}P^m$ the homomorphism

$$\begin{array}{rccll} \alpha(X): & E_x \cong x & \rightarrow & x^\perp = F_x & \\ & u & \mapsto & \Pi_F(\nabla^0_X(u)) & \text{where } \nabla^0_X(u) = \dot{u}(0) \end{array}$$

where $u(t)$ is a curve of $\mathbb{C}^{m+1} - \{0\}$, $u(0) = u$ and $\forall t, u(t) \in x(t)$. For another curve v, we check that $\dot{u}(0) = \dot{v}(0)$ in $\mathbb{C}^{m+1}/x$. We identify $\mathbb{C}^{m+1}/x$ with $x^\perp$ where we use the standard hermitian product h on $\mathbb{C}^{m+1}$ to define the orthogonal.

Using this identification α,

- we can give a definition for the Fubiny-Study metric. Let $X, Y \in T_x\mathbb{C}P^m$,

$$f_x(X, Y) = \Re e\,[h(\alpha(X)(u), \alpha(Y)(u))] = \langle \alpha(X)(u), \alpha(Y)(u) \rangle$$

where u is a unitary generator of x (the expression of this f in local coordinate coincides with the usual one described in [KN69, T. 2, chap. IX, §6,7]).

- let ϵ be a section of E, the connection for the bundle E can be written as:

$$\nabla^E_V \epsilon = \Pi_E(\nabla^0_V \epsilon) = \nabla^0_V \epsilon - \alpha(V)(\epsilon)$$

where ∇^0 is the flat connection in the trivial bundle $\mathbb{C}P^m \times \mathbb{C}^{m+1}$ (so $\nabla^0_V \epsilon$ is the usual derivative $V(\epsilon)$ of the function $\epsilon : \mathbb{C}P^m \rightarrow \mathbb{C}^{m+1}$).

2.2. **Hilbert polynomial.** Let consider H the dual of E. Let γ be a 2-form on $\mathbb{C}P^m$ such that $[\gamma] = c_1(H)$. It is well known that $\int_{\mathbb{C}P^1} \gamma = -1$. We can compute $c_1(M) = -(m+1)[\gamma]$ so $\mathbb{C}P^m$ is spin iff m is odd. We have $c_1(\kappa) = -c_1(M)$ where κ is the canonical bundle (i.e. $\kappa = \wedge^n_{\mathbb{C}} T^*M$). If M is spin, $c_1(\kappa) \equiv 0[2]$ thus there exists a square root for the line bundle κ, which we denote $\kappa^{1/2}$. As $c_1(H) = [\gamma]$, $c_1(\kappa) = [(m+1)\gamma]$ and the line bundle are classified by there first Chern class, we have $H^{\otimes(m+1)} \cong \kappa$ and $H^{\otimes(m+1)/2} \cong \kappa^{1/2}$.

We define and compute the Hilbert polynomial ([LM89, IV §11, p. 365])

$$P(t) := \{e^{\frac{1}{2}t\gamma}\, \widehat{\mathrm{A}}(M)\}[M] = \frac{1}{2^m m!}\prod_{j=1}^{m}(t-m+2j-1)$$

where $\widehat{\mathrm{A}}(M)$ is the $\widehat{\mathrm{A}}$-class of M belonging to the cohomology $H^{4*}(M)$.

For any metric g on $\mathbb{C}P^m$, we let Σ_g be its spin bundle and $D\!\!\!/^g$ is the Dirac operator relative to the metric g. We apply Atitah-Singer index theorem to the twisted Dirac operator $D\!\!\!/^{g+}_{\kappa^{1/2}}$. One of the most important consequences of index theory is the topological invariance of the index which is given by

$$\mathrm{ind}\left(D\!\!\!/^{g+}_{\kappa^{1/2}} : \Sigma_g^+ \otimes \kappa^{1/2} \longmapsto \Sigma_g^- \otimes \kappa^{1/2}\right) = \{\mathrm{ch}(\kappa^{1/2})\, \widehat{\mathrm{A}}(M)\}[M] = P(m+1) = 1$$

where $\mathrm{ch}(\kappa^{1/2}) = e^{c_1(\kappa^{1/2})}$ is the Chern character of the line bundle $\kappa^{1/2}$.

We have also $P(-m-1) = -1$. Therefore the twisted Dirac operator on $\Sigma_g \otimes E^{\otimes(m+1)/2}$ has non-zero harmonic spinors.

2.3. **Complementary bundle for $E^{\otimes(m+1)/2}$.** We remark that, given a classifying map $\phi : X \to \mathbb{C}P^m$ for a line $L = \phi^*E$, we can defined $L^\perp := \phi^*F$, then the bundle $L \oplus L^\perp$ is trivial, that is $L^\perp$ is a complementary bundle for L.

Let $k = \frac{m+1}{2}$. Consider all the homogeneous monomials of degree k with $(m+1)$ variables $X_0, X_1, \cdots X_m$ and more precisely the folowing monomials:

$$M_\alpha(X_0, X_1, \cdots, X_m) = \sqrt{\frac{k!}{\alpha_0! \cdots \alpha_m!}}\, X_0^{\alpha_0} X_1^{\alpha_1} \cdots X_m^{\alpha_m}$$

where $\sum \alpha_i = k$. There are $N+1$ such monomials with $N+1 = \begin{pmatrix} m+k \\ m \end{pmatrix}$.

Now we define the Veronese map of degree k given in homogeneous coordinates by

$$\begin{array}{cccc} \psi_k : & \mathbb{C}P^m & \longrightarrow & \mathbb{C}P^N \\ & [z_0 : \cdots : z_m] & \longmapsto & [M_0(z_0, \cdots, z_m) : \cdots : M_N(z_0, \cdots, z_m)] \end{array}$$

The map ψ_k is an embedding of $\mathbb{C}P^m$ in $\mathbb{C}P^N$. Thanks to the coefficient $\sqrt{\frac{k!}{\alpha_0! \cdots \alpha_m!}}$ (see [MS82, NO76]), we can lift this map to the spheres and, by a computation in local coordinate, we can prove that

$$\psi_k^* f_{\mathbb{C}P^N} = k f_{\mathbb{C}P^m}.$$

Moreover, the map ψ_k classifies $H^{\otimes k} = \psi_k^*(H)$ on $\mathbb{C}P^m$ (see [GH78, chap. 1, §4]). Therefore

$$E_k := \psi_k^*(E) \cong E^{\otimes k}, \quad F_k := \psi_k^*(F) = E^{\otimes k \perp} \quad \text{and} \quad E_k \oplus F_k = \mathbb{C}P^m \times \mathbb{C}^{N+1} =: \mathbb{T}$$

We have also the pull-back connections $\nabla^{E_k} := \psi_k^*(\nabla^E)$ on E_k and $\nabla^{F_k} := \psi_k^*(\nabla^F)$ on F_k. Thus, on $\mathbb{T} = E_k \oplus F_k$, we have two connections $\psi_k^* \nabla^0$ (it is the trivial connection) and $\psi_k^*(\nabla^E \oplus \nabla^F)$.

3. The proof

3.1. **Review on the approach of Vafa-Witten.** As $E_k \cong E^{\otimes(m+1)/2}$, we have already proved, using Hilbert polynomial, that the index of $D\!\!\!/^{g+}_{E_k}$ is non zero and thus the kernel of $D\!\!\!/^{g}_{E_k}$ is non trivial. Let us now consider the tensor product bundle $\Sigma_g \otimes (E_k \oplus F_k)$ endowed with the connection $\nabla^g \otimes 1 + 1 \otimes (\nabla^{E_k} \oplus \nabla^{F_k})$ and denote by $\mathcal{D}^1$ the twisted Dirac operator attached to this connection. Its kernel is also non trivial and there exists a non zero harmonic spinor ϕ_0 of $\mathcal{D}^1$ which lives in the L^2-sections of $\Sigma_g \otimes E_k \subset \Sigma_g \otimes (E_k \oplus F_k) = \Sigma_g \otimes \mathbb{T}$.

Let us now consider the tensor product bundle $\Sigma_g \otimes \mathbb{T}$, endowed this time with the connection $\nabla^g \otimes 1 + 1 \otimes \psi_k^* \nabla^0$. As the pair $(\mathbb{T}, \psi_k^*\nabla^0)$ is a *trivial* flat bundle on $\mathbb{C}P^m$, the spectrum of the twisted Dirac operator $\mathcal{D}^0$ attached to this connection is the same as the spectrum of the Dirac operator $D\!\!\!/^g$ on Σ_g, but with each eigenvalue repeated $N+1$ times its multiplicity.

Let consider the zero order operator $L := \mathcal{D}^0 - \mathcal{D}^1$. For a non-zero harmonic spinor ϕ_0 of $\mathcal{D}^1$, which belongs to the L^2-sections of $\Sigma_g \otimes \psi_k^* E$, we have

$$|\lambda_1(g)| \|\phi_0\|^2 \leq |\langle \mathcal{D}^0 \phi_0, \phi_0 \rangle| = |\langle \mathcal{D}^0 \phi_0 - \mathcal{D}^1 \phi_0, \phi_0 \rangle| \leq \|L\| \|\phi_0\|^2$$

where L is considered as a linear operator from the L^2-sections of $\Sigma \otimes \psi_k^* E$ to the L^2-sections of $\Sigma \otimes \mathbb{T}$. Thus

$$\lambda_1(g)^2 \leq \|L\|^2.$$

3.2. **Computation.** We have

$$\|L\|^2 = \sup_{\|\phi\|=1} \|L(\phi)\|^2 = \sup_{\|\phi\|=1} \int_{\mathbb{C}P^m} \langle L_x(\phi_x), L_x(\phi_x) \rangle \mathrm{dvol}_g \leq \max_{x \in \mathbb{C}P^m} \|L_x\|^2.$$

Thus is suffices to give an upper bound for $\|L_x\|$: the computation is pointwise.

It is easy to compute L_x on a decomposed section $\sigma \otimes \psi_k^* \tau \in \Sigma_g \otimes \psi_k^* E \subset \Sigma_g \otimes \mathbb{T}$. Moreover, it is sufficient because E is of complex rank 1 : in fact, we can write every complex section of $\Sigma_g \otimes E_k$ as $\sigma \otimes \psi_k^* \tau$ where σ is a section of Σ_g and τ is a fixed everywhere non-zero section of E.

We denote the Clifford action relative to g on the bundle Σ_g by $c_g(\cdot)$. If $\{e_a\}_{a=1,\cdots,2m}$ is any g-orthonormal basis on $\mathbb{C}P^m$,

$$\begin{aligned} L_x(\sigma \otimes \psi_k^* \tau) &= \Big[\sum_{a=1}^{2m} c_g(e_a) \big(\nabla^g_{e_a} \sigma \otimes \psi_k^* \tau + \sigma \otimes (\psi_k^* \nabla^0)_{e_a} \psi_k^* \tau \\ &\qquad - \nabla^g_{e_a} \sigma \otimes \psi_k^* \tau - \sigma \otimes (\psi_k^* \nabla^E)_{e_a} \psi_k^* \tau \big) \Big]_x \\ &= \sum_{a=1}^{2m} [c_g(e_a)\sigma]_x \otimes \big[\nabla^0_{\psi_{k*} e_a} \tau - \nabla^E_{\psi_{k*} e_a} \tau \big]_{\psi_k(x)} \\ &= \sum_{a=1}^{2m} [c_g(e_a)\sigma]_x \otimes [\alpha(\psi_{k*} e_a)(\tau)]_{\psi_k(x)} \end{aligned}$$

As $g \geq f$, we can choose a f-orthonormal basis $\{\varepsilon_a\}$ and a g-orthonormal basis $\{e_a\}$ such that $e_a = \mu_a \varepsilon_a$ with $\mu_a \in]0, 1]$. Moreover, as the metric g is hermitian, we can suppose that $J e_a = e_{m+a}$ for $a \in \{1, \cdots, m\}$. Thus $\mu_{m+a} = \mu_a$.

We use the hermitian metric induced by g on Σ_g and the hermitian metric h coming from $\mathbb{C}^{N+1}$ on $\mathbb{T}$. We have:

$$\|L_x(\sigma\otimes\psi_k^*\tau)\|^2 = \sum_{a,b=1}^{2m}\langle c_g(e_a)\sigma, c_g(e_b)\sigma\rangle_{g,x}\cdot h(\alpha(\psi_{k*}(e_a))(\tau),\alpha(\psi_{k*}(e_b))(\tau))_{\psi_k(x)}$$

A short computation shows that $\langle c_g(e_a)\sigma, c_g(e_a)\sigma\rangle_{g,x} = |\sigma|_g^2$ and $\langle c_g(e_a)\sigma, c_g(e_b)\sigma\rangle_{g,x}$ is a pure imaginary complex if $a\neq b$, thus

$$\begin{aligned}\|L_x(\sigma\otimes\psi_k^*\tau)\|^2 &= \sum_{a=1}^{2m}|\sigma|_{g,x}^2\cdot h(\alpha(\psi_{k*}(e_a))(\tau),\alpha(\psi_{k*}(e_a))(\tau))_{\psi_k(x)}\\ &\quad+\sum_{a\neq b}\langle c_g(e_a)\sigma, c_g(e_b)\sigma\rangle_{g,x}\cdot i\Im m\left[h(\alpha(\psi_{k*}(e_a))(\tau),\alpha(\psi_{k*}(e_b))(\tau))_{\psi_k(x)}\right]\end{aligned}$$

Using the definition of the Fubini-Study metric in terms of α, the equality $\psi_k^* f^{\mathbb{C}P^N} = kf^{\mathbb{C}P^m}$ and the fact that $\{\frac{e_a}{\mu_a}\} = \{\epsilon_a\}$ is an orthonormal basis for $f^{\mathbb{C}P^m}$, we prove that:

$$h(\alpha(\psi_{k*}(e_a))(\tau),\alpha(\psi_{k*}(e_a))(\tau))_{\psi_k(x)} = \mu_a^2 k|\tau|^2$$

and

$$\Im m h(\alpha(\psi_{k*}(e_a))(\tau),\alpha(\psi_{k*}(e_b))(\tau))_{\psi_k(x)} = \begin{cases}-\mu_a^2 k|\tau|^2 \text{ if } a\in\{1,\cdots,m\} \text{ and } b=m+a\\ +\mu_b^2 k|\tau|^2 \text{ if } b\in\{1,\cdots,m\} \text{ and } a=m+b\\ 0 \text{ in the other cases}\end{cases}$$

Thus, using $\mu_a = \mu_{m+a}$,

$$\begin{aligned}\|L_x(\sigma\otimes\psi_k^*\tau)\|^2 &= 2k|\sigma|^2|\tau|^2\sum_{a=1}^m \mu_a^2\\ &+ k|\tau|^2\left[-\sum_{a=1}^m i\mu_a^2\langle c_g(e_a)\sigma, c_g(Je_a)\sigma\rangle_{g,x}+\sum_{b=1}^m i\mu_b^2\langle c_g(Je_b)\sigma, c_g(e_b)\sigma\rangle_{g,x}\right]\\ &= k|\tau|^2\left[2\sum_{a=1}^m\mu_a^2|\sigma|^2+2i\langle c_g(\Omega)\sigma,\sigma\rangle\right]\end{aligned}$$

where Ω is the Kähler form relative to f and $c_g(\Omega) := -\sum_{a=1}^m \mu_a^2 c_g(e_a)c_g(Je_a)$.

We check that

$$\|c_g(\Omega)^2\| \le (\sum_{a=1}^m \mu_a^2)^2, \quad \text{thus} \quad \|c_g(\Omega)\| \le \sum_{a=1}^m \mu_a^2.$$

As $\mu_a \le 1$ and $k=(m+1)/2$, we conclude that

$$\|L\|^2 \le 4k\sum_{a=1}^m \mu_a^2 \le 2m(m+1).$$

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Hélène Davaux, Laboratoire UMPA, 46 allée d'Italie, 69364 LYON, France
E-mail address: hdavaux@umpa.ens-lyon.fr

Maung Min-Oo, Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario L8S 4K1, Canada
E-mail address: minoo@univmail.cis.mcmaster.ca