A remark about dihedral group actions on spheres

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ABSTRACT

We show that a finite dihedral group does not act pseudofreely and locally linearly on an even-dimensional sphere \( S^{2k} \), with \( k > 1 \). This answers a question of Kulkarni from 1982.

1. Introduction

In this note, we let \( D_p = \langle a, b \mid a^p = b^2 = 1, \ bab = a^{-1} \rangle \) denote the finite dihedral group of order \( 2p \), for \( p \) an odd prime. A famous theorem of Milnor \cite{8} states that a finite dihedral group cannot act freely on a topological \( n \)-manifold with the mod 2 homology of \( S^n \). More generally, a pseudofree action is one which is free outside of a discrete set of points. In \cite[Theorem 7.4]{6}, Kulkarni studied orientation-preserving, pseudofree actions of finite groups \( G \) on manifolds which are \( \mathbb{Z}/2 \)-homology \( n \)-spheres, and found that, for \( n = 2k \), the group \( G \) must be (i) a periodic group which acts freely on \( S^{2k-1} \), (ii) dihedral, or (iii) tetrahedral, octahedral, or icosahedral (when \( k = 1 \)). The first case occurs as the suspension of any free action of a periodic group on \( S^{2k-1} \), and the other cases already appear for orthogonal actions on \( S^2 \). Kulkarni asked whether the second case could actually occur on \( S^{2k} \) if \( k > 1 \). This turns out to be impossible.

**Theorem A.** The dihedral group \( G = D_p \), with \( p \) an odd prime, cannot act pseudofreely and locally linearly on \( S^{2k} \), preserving the orientation, for \( k > 1 \).

For \( k \) even, we show that there does not even exist a finite pseudofree \( G \)-CW complex \( X \simeq S^{2k} \), with \( X^G = \emptyset \). For all odd integers \( k \geq 1 \), such complexes do exist, for example, by taking the join of \( S^2 \) with the action given by \( G \subset SO(3) \) and a finite Swan complex for \( G \) (see \cite{3, 9}).

**Remark 1.1.** My interest in this question was prompted by the recent paper of Edmonds \cite{2}, where he proves this result for \( k \) even. Our methods seem rather different. The discussion by Edmonds in \cite[4.1]{2} combined with Theorem A shows that there are no effective pseudofree dihedral actions on \( S^n \), for \( n > 2 \), even if some elements of \( G \) are allowed to reverse orientation.

2. The chain complex

In this section, we let \( G = D_p \) and suppose that \( X \) is a finite \( G \)-CW complex such that \( X \simeq S^{2k} \), with \( k > 0 \), and \( X^G = \emptyset \). We further assume that the \( G \)-action is pseudofree and induces the identity on homology. It follows from \cite[Proposition 7.3]{6} that every non-identity element of \( G \) fixes exactly two points. We assume that \( X^G = \emptyset \) since this is a necessary condition for a locally...
linear, pseudo-free action on a sphere (by Milnor’s theorem). Let $C = C(X^2)$ denote the chain complex of $X$ over the orbit category $\mathbb{Z} \Gamma := \mathbb{Z} \text{Or}_\mathcal{F} G$ with respect to the family $\mathcal{F}$ of all proper subgroups of $G$ (see [1] or [7] for this theory). The notation means that $C_\iota(G/U) = C_\iota(X^U)$, for $U \leq G$, and the action of $N_G(U)/U$ on $C_\iota(G/U)$ induced by the $G$-action on $X$ is expressed algebraically through the functorial properties of $C$.

Our pseudofree assumption on the $G$-CW complex $X$ implies that $C_\iota(G/U) = 0$, if $U \neq 1$ is a non-trivial subgroup of $G$, and $i > 0$. Therefore,

$$H_i(C)(G/U) = 0 \quad \text{if } i > 0, \text{ for all } U \neq 1. \quad (1)$$

From the homology of $S^{2k}$ we have

$$H_0(C)(G/1) = \mathbb{Z} \quad \text{and} \quad H_i(C)(G/1) = 0 \quad \text{for } i \neq 0, 2k. \quad (2)$$

In addition, since we assumed that $G$ acts trivially on the homology of $S^{2k}$, we have

$$H_{2k}(C)(G/1) = \mathbb{Z}, \quad \text{with trivial } G\text{-action.} \quad (3)$$

Let $H = \langle a \rangle$ and $K = \langle b \rangle$ denote particular subgroups of $G$, of order $p$ and $2$, respectively. The orbit types give the chain group

$$C_0 = \mathbb{Z}[G/H^?] \oplus \mathbb{Z}[G/K^?] \oplus \mathbb{Z}[G/K^?],$$

where $\mathbb{Z}[G/V^?]$ denotes the free right module over the orbit category with values

$$\mathbb{Z}[G/V^?](G/U) = \mathbb{Z} \text{Map}_G(G/U, G/V),$$

for all proper subgroups $U \leq G$. In particular, the homology of the fixed sets is given by

$$H_0(C)(G/H) = \mathbb{Z}[G/H^?](G/H) = \mathbb{Z}[N_G(H)/H] = \mathbb{Z}[\mathbb{Z}/2] \quad (4)$$

and

$$H_0(C)(G/K) = (\mathbb{Z}[G/K^?](G/K))^2 = (\mathbb{Z}[N_G(K)/K])^2 = \mathbb{Z} \oplus \mathbb{Z}. \quad (5)$$

**Definition 2.1.** A finite $\mathbb{Z} \Gamma$-chain complex $C$ of finitely generated free $\mathbb{Z} \Gamma$-modules, which satisfies the algebraic conditions (1)–(5), is called a pseudofree $\mathbb{Z} \Gamma$-chain complex with the $\mathbb{Z}$-homology of $S^{2k}$.

One example of such a complex arises from the standard orthogonal action $Y = S(V)$ of the dihedral group on $S^2$ (for $G$ as a subgroup of $\text{SO}(3)$). The $\text{SO}(3)$-representation $V = W \oplus \mathbb{R}_-$ is the sum of the standard 2-dimensional real representation $W$ (given by the action on a regular $2p$-gon in the plane) and the non-trivial 1-dimensional real representation $\mathbb{R}_-$. The chain complex $D = C(Y^?)$ over the orbit category has the form

$$
\begin{array}{c}
\mathbb{Z}[G/1^?] \ar[r]^{3} & \mathbb{Z}[G/1^?]^3 & \mathbb{Z}[G/H^?] \oplus (\mathbb{Z}[G/K^?]^2)
\end{array}
$$

where $H_2(D) = \mathbb{Z}_0$ is the $\mathbb{Z} \Gamma$-module with value $\mathbb{Z}_0(G/1) = \mathbb{Z}$, and zero otherwise. The module $H_0 := H_0(D)$ has value $H_0(G/1) = \mathbb{Z}$, and values at $G/H$ and $G/K$ as listed above. In general, for any pseudofree $\mathbb{Z} \Gamma$-chain complex $C$ with the $\mathbb{Z}$-homology of $S^{2k}$, we have $H_{2k}(C) = \mathbb{Z}_0$ and $H_0(C) = H_0(D)$.

**Lemma 2.2.** Suppose that $C$ is a pseudofree $\mathbb{Z} \Gamma$-chain complex with the $\mathbb{Z}$-homology of $S^{2k}$. Then the complex $C$ is chain homotopy equivalent to a finite free $2k$-dimensional chain
complex $C'$, with $C'_i = C_i$ for $i \geq 4$, whose initial part $C'_2 \to C'_1 \to C'_0$ is chain isomorphic to $D$.

Proof. Since $H_0(C) = H_0(D)$, this follows from the version of Schanuel’s lemma over the orbit category given in the proof of [4, Lemma 8.12].

An immediate consequence is the statement of Theorem A for $k$ even.

**Corollary 2.3 (Edmonds [2]).** Let $G = D_p$. If $k$ is even, there is no effective pseudofree $G$-action on a finite $G$-CW complex $X \simeq S^{2k}$, inducing the identity on homology.

Proof. Let $C = C(X^?)$ denote the chain complex over the orbit category of such an action. From the chain equivalent complex $C' \simeq C$, we can extract a periodic resolution

$$0 \longrightarrow \mathbb{Z}_0 \longrightarrow C_{2k} \longrightarrow C_{2k-1} \longrightarrow \cdots \longrightarrow C_4 \longrightarrow C'_3 \longrightarrow \mathbb{Z}_0 \longrightarrow 0$$

since $H_2(D) = H_{2k}(C) = \mathbb{Z}_0$, where $C'_3$ is a direct sum of copies of $\mathbb{Z}[G/1^?]$. By evaluating at $G/1$, we obtain a periodic projective resolution from $\mathbb{Z}$ to $\mathbb{Z}$ over $\mathbb{Z}G$ of length $(2k - 2)$. Since $G = D_p$ has periodic cohomology of period 4 (and not 2), we conclude that $k$ is odd.

Proof of Theorem A ($k$ odd). Suppose, if possible, that we have a locally linear and orientation-preserving pseudofree topological action of $G$ on $S^{2k}$, for some odd integer $k \geq 3$. Then there exists a finite $G$-CW complex $X \simeq S^{2k}$, and a chain homotopy equivalence $C(X^?) \simeq C'$ provided by Lemma 2.2. We may identify the singular set $\text{Sing}(X)$ of $X$ with the singular set of the given action on $S^{2k}$. Let $\{x_0, x_1, x_2\} \subset \text{Sing}(X)$ denote representatives of the distinct $G$-orbits of singular points (with $G_{x_0} = H$, and $G_{x_i} = K$ for $i = 1, 2$). Around each singular point $x_i$, with $0 \leq i \leq 2$, we can choose a linearly embedded 2-disk slice $G \times_{G_{x_i}} D^2 \subset S^{2k}$, since the action $(S^{2k}, G)$ is locally linear. This gives a $G$-equivariant embedding

$$f_0 : \bigcup_{0 \leq i \leq 2} (G \times_{G_{x_i}} D^2) \subset S^{2k}.$$ 

Since the pseudofree orbit structure of the standard $G$-action on $S^2 = S(V)$ is the same for any locally linear action on $S^{2k}$, we can consider $f_0$ to be a $G$-equivariant embedding of a tubular neighborhood of the singular set of $S(V)$ into $S^{2k}$. By obstruction theory, and since $k \geq 3$, we can extend this embedding $f_0$ to a $G$-equivariant embedding $f : S(V) \subset S^{2k}$. Non-equivariantly such an embedding of $S^2 \subset S^{2k}$ is isotopic to a standard embedding. We have thus obtained a dihedral action on $S^{2k}$ of the type considered in my earlier joint work with Pedersen [5], namely, one conjugate to ‘a topological action on a sphere which is free off a standard proper subsphere, and given by a $S(V)$ on the subsphere’. However, we proved in [5, Theorem 7.11] that such an action exists if and only if the representation $V$ on the subsphere contains two $\mathbb{R}$-factors. Since this is not the case for the standard $SO(3)$-representation $V$ of $G$, we conclude that a pseudofree $G$-action on $S^{2k}$ does not exist for $k > 1$.

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References


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