# Transient Behaviour of Batch Queues: A Combinatorial Approach

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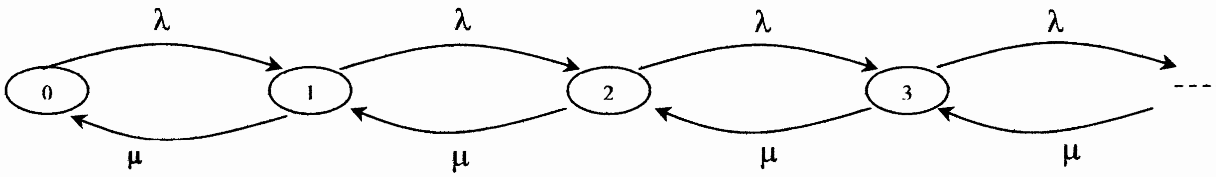
## 1.Introduction

Model M/M/1:
One server;
Arrivals are according to Poisson process with rate $\lambda$;
Service times independent of arrivals are iid exponential with mean $1/\mu$.

M/M/1 queue with the following state rate diagram:




**Notations:**
Q(t): Queueing Process at time t
$P_{jk}(t)$: $P(Q(t) = k | Q(0) = j)$
$P_n(t)$: $P(Q(t)=n)$

Transient behaviour: $P_n(t)$
Steady-state behaviour: lim $P_n(t)$ as t tends to infinity

$$\text{Rate Matrix } Q = \begin{bmatrix} -\lambda & \lambda & 0 & . \\ \mu & -(\lambda+\mu) & \lambda & . \\ 0 & \mu & -(\lambda+\mu) & . \\ . & . & . & . \end{bmatrix}$$

**Classical method of solution for transient behaviour (Bailey(1954))**

The Kolmogorov difference-differential equations are:

$$\frac{dP_n(t)}{dt} = \mu P_{n+1}(t) + \lambda P_{n-1}(t) - (\lambda + \mu) P_n(t) \qquad n \geq 1$$

$$\frac{dP_0(t)}{dt} = \mu P_1(t) - \lambda P_0(t)$$

=> Note: ***d/dt (P(t)) = P(t) . Q***

Use PGF on n and LT on t and finally get

$$P_{jk}(t) = e^{-(\lambda+\mu)t}\left[\rho^{(k-j)/2} I_{k-j}\left(2\sqrt{\lambda\mu t}\right) + \rho^{(k-j-1)/2} I_{k+j+1}\left(2\sqrt{\lambda\mu t}\right) + (1-\rho)\rho^{n} \sum_{r=k+j+2}^{\infty} \rho^{-r/2} I_r\left(2\sqrt{\lambda\mu t}\right)\right] \qquad (1)$$

where

$\rho = \lambda/\mu$

$I_u(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{u+2r}}{r!(u+r)!}$ is the modified Bessel function of the first kind.

Note: Steady state solution $P_j = (1 - \rho)\, \rho^j$, $\rho < 1$

**Random walk approach (Champernowne (1956))**

Assume there are $a$ arrivals and $b$ virtual service completions in time $t$. Then each event is an arrival with probability $\lambda/(\lambda+\mu)$ and a service completion with probability $\mu/(\lambda+\mu)$. This defines a random walk with +1 and -1 steps, which is not allowed to go below 0. Eventually, the solution involves counting certain random walk paths. What Champernowne did was to tally his solution with the classical one.

Champernowne's random walk approach to combinatorial solution:

$$P_{jk}(t) = e^{-(\lambda+\mu)t}\left(\sum_{b=0}^{\infty}\frac{1}{b!(b+k-j)!} - \sum_{b=j}^{\infty}\frac{1}{(b-j)!(b+k)!}\right)\lambda^{b+k-j}\mu^{2b+k-j}$$

$$+\sum_{b=j}^{\infty}\sum_{r=0}^{b-j}\left(\frac{1}{(b-j-r)!(b+k)!} - \frac{1}{(b-j-r-1)!(b+k+1)!}\right)\lambda^{b+k-j-r}\mu^{b}t^{2b+k-j-r}$$

(2)

Note we have used $k = a + j - b$.

(Ref: Jain, Mohanty, Böhm (2006) " A Course on Queueing Models")

## 2. Preliminaries

Randomization Theorem.

Suppose a Markov process on a countable state space has transition rate matrix **Q** with sup $|q_{ii}| \leq c < \infty$. Then the transition probability function may be written as

$$P_{ij}(t) = e^{-ct} \sum_{n=0}^{\infty} \frac{(ct)^n}{n!} P_{ij}^{(n)} \quad \text{for } i,j = 0,1.2,... \quad (\#)$$

where $P_{ij}^{(n)}$ is the n-step transition probability in the associated Markov chain which has the stochastic matrix

$$P = \frac{1}{c} Q + I, \qquad (\#\#)$$

as its 1-step transition probability matrix, $I$ being the identity matrix.

$$\text{In our case, } c = \lambda + \mu \text{ and therefore } P = \begin{bmatrix} q & p & 0 & . \\ q & 0 & p & . \\ 0 & q & 0 & . \\ . & . & . & . \end{bmatrix}$$

$$\text{with } p = \frac{\lambda}{\lambda + \mu}, \; q = \frac{\mu}{\lambda + \mu} \; .$$

Dual Process (Anderson (1991)).

For a Markov process X(t) with its transition rate matrix **Q**, its associated dual process X*(t) with **Q*** is defined by

$$q_{ij}^{*} = \sum_{k=i}^{\infty} \left( q_{jk} - q_{j-1,k} \right) \qquad (*)$$

for $i,j = 0,1,2,..$ where we assume $q_{-1,k} = 0$ for every $k$. It exists, provided Q* is a rate matrix.

Duality Theorem (Anderson (1991)).

Suppose $P_{ij}(t)$ is the transition probability function of the Markov Process X(t). Define

$$P^*_{ij}(t) = \sum_{k=i}^{\infty}\left[P_{jk}(t) - P_{j-1,k}(t)\right]$$

for states $i, j = 0,1,2...$ with the convention that $P_{-1,k}(t) = 0$. Then $P^*_{ij}(t)$ is the unique transition probability function associated with **Q*** if and only if $P_{ij}(t)$ is stochastically monotone.

Note: $P_{ij}(t)$ is stochastically monotone if $\sum_{j>k+1} P_{ij}(t)$ is an increasing function of i for every fixed k and t. This says that the chance of ending up in the tail region is higher as i becomes larger.

Moreover,

$$P_{ij}(t) = \sum_{k=0}^{i}\left[P^*_{jk}(t) - P^*_{j+1,k}(t)\right] \quad \text{for i,j = 0,1,2,...} \qquad (**)$$

**Algorithm:**

Q => Q* by (*)

Q* => P* by (##), associated Markov chain

Determine $P^{*(n)}_{i,j}$ ----- This part is done by lattice path combinatorics

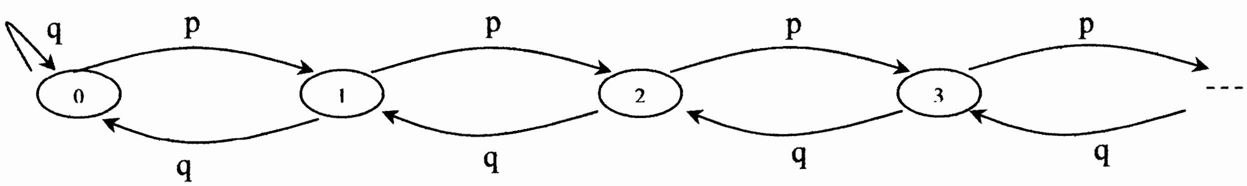
$P^*_{ij}(t)$ <= $P^{*(n)}_{i,j}$ by Randomization Th. (#)

$P_{ij}(t)$ <= $P^*_{ij}(t)$ by (**)

## 3. M/M/1 Revisited

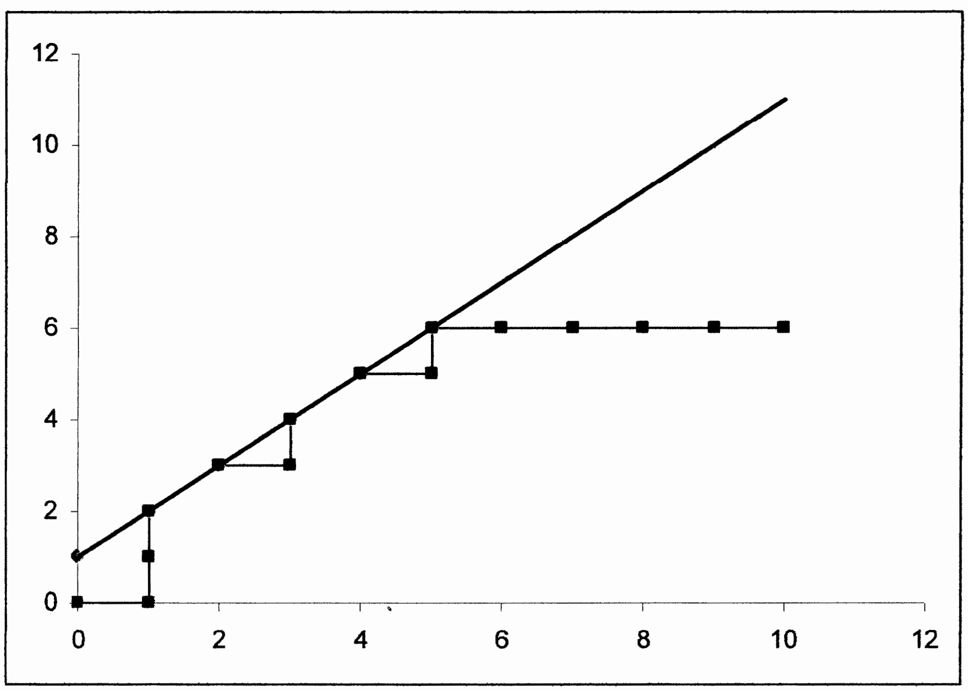
Without duality; Use Randomization Theorem and determine $P_{ij}^{(n)}$.

State transition probability diagram of the Markov chain:




Lattice path combinatorics

Represent a forward step by a horizontal (x) unit and a backward step by a vertical (y) unit. Then a realization of the chain is represented by a lattice path. A chain starting from i is a lattice path from the origin not crossing the line y = x +i and having diagonal steps on y =x+i.




The above diagram is a lattice path representation of a queueing sequence having $i=1$ unit initially, $a=8$ arrivals and $b=6$ virtual service completions.

Counting result 1 (Mohanty(1979))

*# lattice paths from the origin to (a,b) not touching the line y = x +j is*

$$\binom{a+b}{b} - \binom{a+b}{b-j}. \qquad (3)$$

The first term is the total number of paths. The number of those which touch or cross the line is obtained by the reflection principle and is given by the second term.

Use this result and take into account of touching the line and diagonal steps.

$$P_{i,j}(t) = e^{-(\lambda+\mu)t}\left\{ \sum_{n=|j-i|}^{j+i} \binom{n}{l} \lambda^{j-i+l} \mu^{l} \frac{t^n}{n!} \right.$$

$$+ \sum_{n=j+i+1}^{\infty} \left( \binom{n}{l} - \binom{n}{l-i-1} \right) \lambda^{j-i+l} \mu^{l} \frac{t^n}{n!}$$

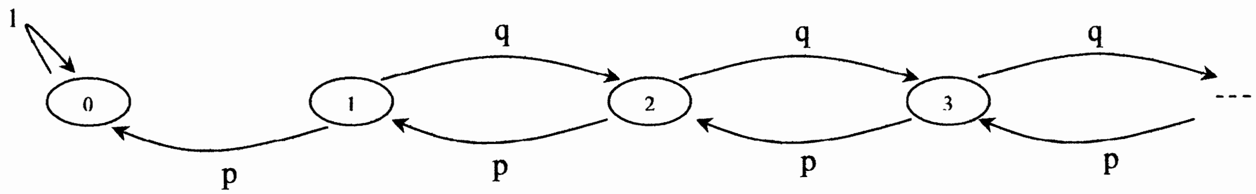
$$\left. + \sum_{n=j+i+1}^{\infty} \sum_{r=i}^{\left[\frac{2l-1}{2}\right]} \left( \binom{n}{r-i} - \binom{n}{r-i-1} \right) \lambda^{j-i+r} \mu^{n-j+i-r} \frac{t^n}{n!} \right\}$$

where $l = (n-j+i)/2$ . (4)

Notes:
1. This result can be obtained from (2) by setting a+b=n.
2. Complications in counting arise out of the fact that state 0 is not an absorbing barrier.

Dual process approach

State transition probability diagram of the associated dual Markov chain:




Observe that state 0 is an absorbing barrier.

Represent the lattice path the same way. Let there be d vertical steps. This time it is a path from the origin to (d-j+k, d) which does not touch the line y=x+j.

Here n = 2d-j+k, => d = (n-k+j)/2.

Use (3) to get the number of paths to be

$$\left[\binom{n}{\frac{n-k+j}{2}}-\binom{n}{\frac{n-j-k}{2}}\right]$$

and hence

$$P^{*(n)}_{j,k}=\left[\binom{n}{\frac{n-k+j}{2}}-\binom{n}{\frac{n-j-k}{2}}\right]p^{\frac{n-k+j}{2}}q^{\frac{n+k-j}{2}}$$

for j =0,1,…, k= 1,2,…Thus

$$P^{*}_{j,k}(t)=e^{-(\lambda+\mu)t}\sum_{n=0}^{\infty}\frac{((\lambda+\mu)t)^n}{n!}P^{*(n)}_{j,k} \qquad \text{for } j=0,1,\ldots, k=1,2,\ldots \tag{5}$$

For k=0, use the matrix form of Kolmogorov equation.

$$\frac{d}{dt}P^*_{j,0}(t) = \lambda P^*_{j,1}(t)$$

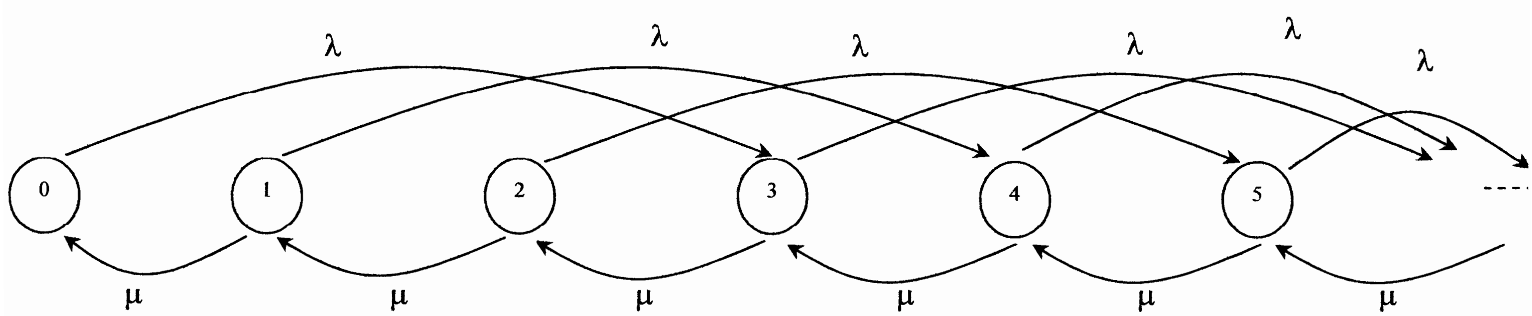
where $P^*_{j,1}(t)$ is given in (5).

Finally we can get $P_{ij}(t)$ through

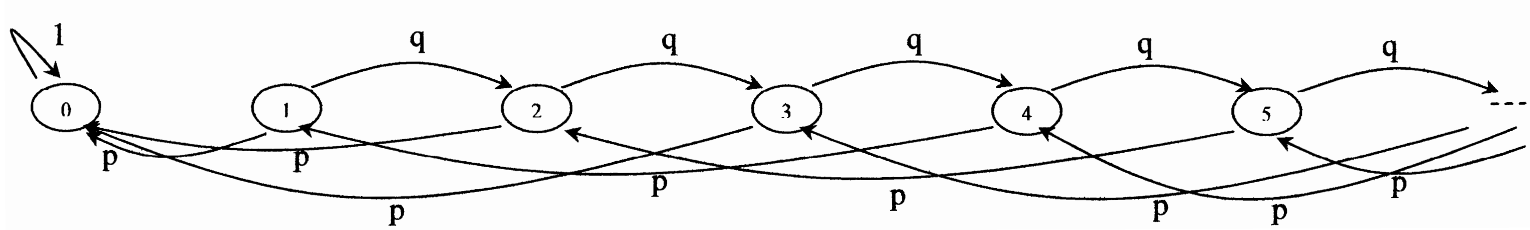
$$P_{i,j}(t) = \sum_{k=0}^{i}\left[P^*_{j,k}(t) - P^*_{j+1,k}(t)\right] = \left[P^*_{j,0}(t) - P^*_{j+1,0}(t)\right] + \sum_{k=1}^{i}\left[P^*_{j,k}(t) - P^*_{j+1,k}(t)\right] \quad (6)$$

## 4. Batch Arrivals

Batch size = H
Rate flow diagram for H=3:




State transition probability diagram of the associated dual Markov chain:



The structure is a random walk with forward steps of size 1 and backward steps of size 3, having an absorbing barrier at 0.

Represent a forward step by a horizontal (x) unit and a backward step by a vertical (y) unit. The sequence is then represented by a lattice path.

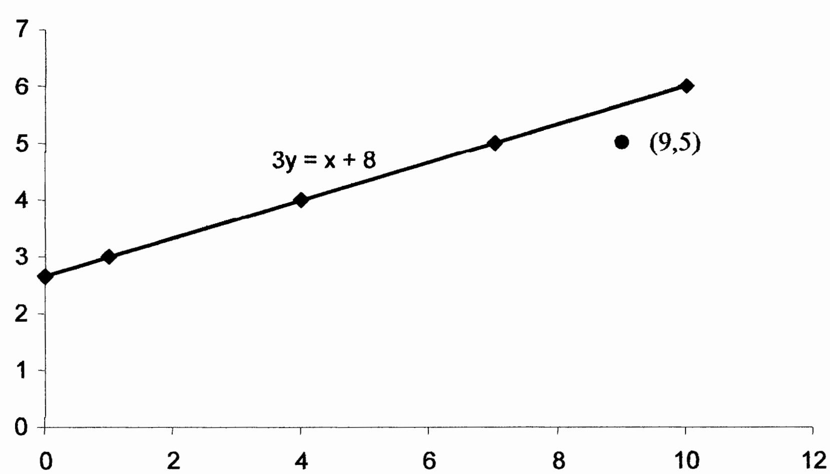
Horizontal unit ⇔ arrival of 1 unit
Vertical unit ⇔ service completion of 3 units
Consider a sequence from i to j in n steps. Let d be the number of backward steps.

⇒ # y units = d
# x units = 3d – i + j
n = 4d – i + j

A random walk sequence ⇔ A path from (0,0) to (3d – i + j, d) without touching the line 3y = x +i, each having the probability $q^{3d-i+j}p^d$.

To count these paths.

For i=8, j=2 and d=5, see the diagram



#paths = # all paths – #paths that touch or cross the line 3y = x + i. Paths can touch or cross only at the marked points.

$$\# \text{ all paths} = \binom{4d-i+j}{d}$$

A general point which a path touches for the first time on the boundary when coming from the end is (3s – i, s), i/3 ≤ s ≤ d. From then on it can reach (0,0) without any constraint.

<u>Counting result 2 (Mohanty (1979))</u>

*# paths from the (0,0) to (a, b) without touching the line x = μy except at the beginning*

$$= \frac{a-\mu b}{a+b}\binom{a+b}{a}$$

Using this, # paths from (3s – i, s) to (3d – i + j, d) without touching the boundary

$$= \frac{j}{4d-4s+j}\binom{4d-4s+j}{d-s}.$$

# paths from the origin to (3s – i, s)

$$= \binom{4s-i}{s}$$

⇒ # desired paths

$$= \binom{4d-i+j}{d} - \sum_{s\geq \frac{i}{3}}^{d}\binom{4s-i}{s}\frac{j}{4d-4s+j}\binom{4d-4s+j}{d-s}$$

We change s to s' by s'=d – s, and then set s'=s, which means the paths are seen in reverse order. Then the above becomes

$$\binom{4d-i+j}{d} - \sum_{s=0}^{d-i/3} \frac{j}{j+4s}\binom{j+4s}{s}\binom{4(d-s)-i}{d-s}$$

For general H, j =0,1,…, and k=1,2,…

$$P^{*}_{j,k}(t) = e^{-(\lambda+\mu)t}\sum_{n=0}^{\infty}\frac{t^{n}}{n!}$$

$$\left[\binom{n}{\frac{n+j-k}{H+1}} - \sum_{s=0}^{\frac{n+j-k}{H+1}-\frac{j}{H}} \frac{k}{k+(H+1)s}\binom{k+(H+1)s}{s}\binom{n-k-(H+1)s}{\frac{n+j-k}{H+1}-s}\right]\lambda^{\frac{n+j-k}{H+1}}\mu^{\frac{Hn-j+k}{H+1}} \quad (7)$$

For j > 0,

$$P^{*}_{j,0}(t) = \left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\left\{\left[1-e^{-(\lambda+\mu)t}\cdot\sum_{r=1}^{n+1}\frac{[t(\lambda+\mu)]^{n-r+1}}{(n-r+1)!}\right]\cdot\sum_{k=1}^{H}P^{*(n)}_{j,k}\right\}. \quad (8)$$

Substitute (7) and (8) in (6) to give $P_{ij}(t)$.

Separate off terms independent of t and get

$$P_{i,j}(t)=\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\sum_{k=1}^{H}\left[P^{*(n)}_{j,k}-P^{*(n)}_{j+1,k}\right]-$$

$$\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\left\{\left[e^{-(\lambda+\mu)t}\cdot\sum_{r=1}^{n+1}\frac{\left[t(\lambda+\mu)\right]^{n-r+1}}{(n-r+1)!}\right]\cdot\sum_{k=1}^{H}\left(P^{*(n)}_{j,k}-P^{*(n)}_{j+1,k}\right)\right\}+$$

$$e^{-(\lambda+\mu)t}\sum_{k=1}^{i}\left[\sum_{n=0}^{\infty}\frac{\left[t(\lambda+\mu)\right]^{n}}{n!}\left(P^{*(n)}_{j,k}-P^{*(n)}_{j+1,k}\right)\right]. \qquad (9)$$

$$\sum_{n=0}^{\infty}P^{*(n)}_{j,k}=\delta_{jk}+\frac{\rho_{jk}}{1-\rho_{kk}}<\infty$$

where $\rho_{jk}$ denotes the probability of starting at state $j$ and eventually reaching state $k$ and $\delta_{jk}=\begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases}$.

In $\lim_{t\to\infty}P_{i,j}(t)$, the last two terms vanish. Therefore,

$$\lim_{t\to\infty}P_{i,j}(t)=\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{k=1}^{H}\left[\left(\delta_{jk}-\delta_{(j+1)k}\right)+\left(\frac{\rho_{jk}-\rho_{(j+1)k}}{1-\rho_{kk}}\right)\right] \qquad (10)$$

This expression either equals $\pi_j$, the steady state distribution of the batch arrival process pictured in Figure 6 (if it exists), or is 0.

Combinatorial expressions for $\rho_{jk}$ s are available, but $\pi_j$ is messy.

$$P_{i,0}(t) =$$

$$1-\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty}\sum_{k=1}^{H} p_{1,k}^{*(n)}+\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty} e^{-(\lambda+\mu)t}\cdot\sum_{r=1}^{n+1}\frac{\left[t(\lambda+\mu)\right]^{n-r+1}}{(n-r+1)!}\cdot\sum_{k=1}^{H} p_{1,k}^{*(n)}$$

$$-e^{-(\lambda+\mu)t}\sum_{k=1}^{i}\left[\sum_{n=0}^{\infty}\frac{\left[t(\lambda+\mu)\right]^{n}}{n!}p_{1,k}^{*(n)}\right] \qquad (11)$$

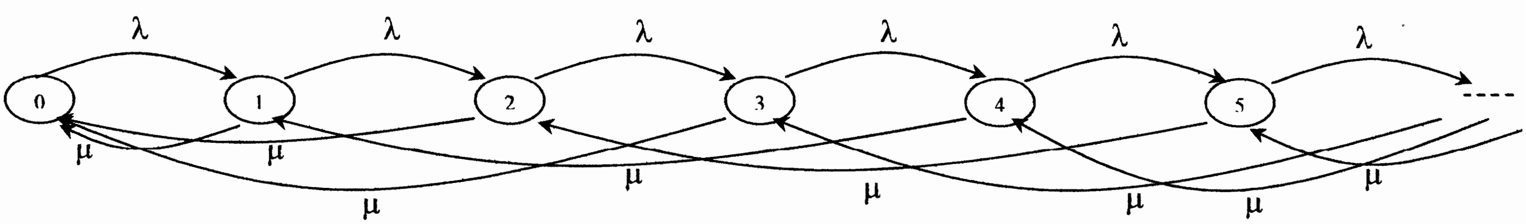
where the independent terms provide the steady state solution.

$$\pi_0 = 1-\frac{\lambda}{\lambda+\mu}\sum_{k=1}^{H}\left(\delta_{1k}+\frac{\rho_{1k}}{1-\rho_{kk}}\right), \text{ if it exists.} \qquad (12)$$

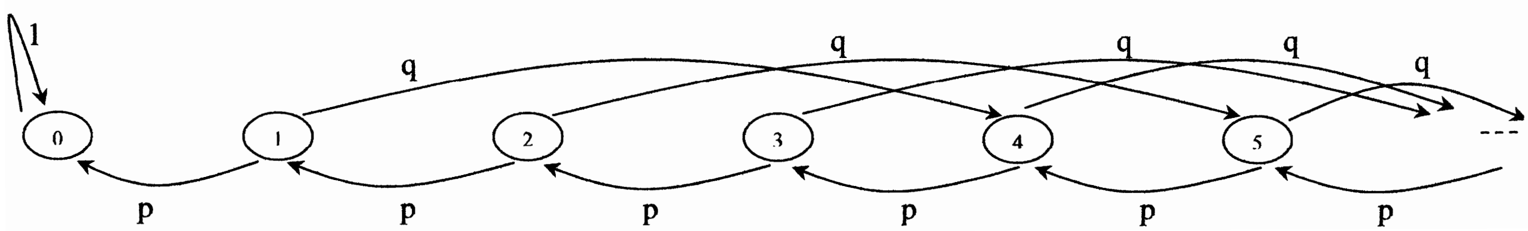
## 5. Batch Services

Batch size = H. If there is less in the queue, the server serves all of them.

Rate flow diagram when H=3:




State transition probability diagram of the dual Markov chain:

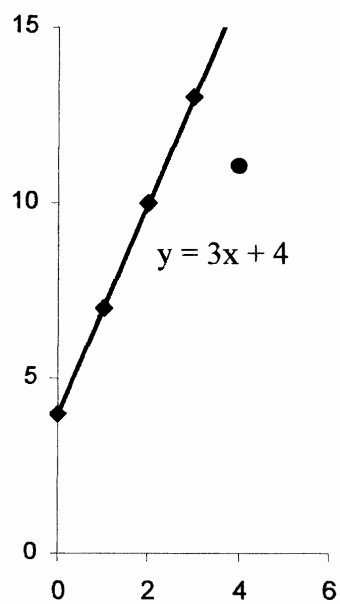


This is almost a reverse situation of the previous one.
Represent a forward step (of size 3) by a horizontal unit and a backward step (of size 1) by a vertical unit.
Let u be the number of forward steps.

⇒ # x units = u
# y units = 3u+i − j
n = 4u + i − j

A sequence ⇔a path from the origin to (u, 3u + i − j) without touching the line y = 3x + i.
To count paths from (0,0) to (u, 3u + i − j) without touching the line y = 3x+i.

For i=4, j=5 and u=4, see the diagram




$$\#\text{ paths} = \binom{4u+i-j}{u} - \sum_{s=0}^{u-\frac{j}{3}} \frac{i}{i+4s}\binom{i+4s}{s}\binom{4(u-s)-j}{u-s}$$

$$P^*_{j,k}(t) = e^{-(\lambda+\mu)t} \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$\left[ \begin{pmatrix} n \\ \frac{n+k-j}{H+1} \end{pmatrix} - \sum_{s=0}^{\frac{n+k-j}{H+1} - \frac{k}{H}} \frac{j}{j+(H+1)s} \begin{pmatrix} j+(H+1)s \\ s \end{pmatrix} \begin{pmatrix} n-j-(H+1)s \\ \frac{n+k-j}{(H+1)} - s \end{pmatrix} \right] \mu^{\frac{n+k-j}{H+1}} \lambda^{\frac{Hn-k+j}{H+1}} \qquad (13)$$

Compare (7) and (13). One is obtained from the other by interchanging j, k and λ, μ. Dual process is usually defined by interchanging input and service distributions.

We also establish for j >0,

$$P^*_{j,0}(t) = \left(\frac{\lambda}{\lambda+\mu}\right) \sum_{n=0}^{\infty} \left\{ \left[ 1 - e^{-(\lambda+\mu)t} \cdot \sum_{r=1}^{n+1} \frac{[t(\lambda+\mu)]^{n-r+1}}{(n-r+1)!} \right] \cdot P^{*(n)}_{j,1} \right\}.$$

=>

$$P_{i,j}(t) = \left(\frac{\lambda}{\lambda+\mu}\right) \sum_{n=0}^{\infty} \left[ P^{*(n)}_{j,1} - P^{*(n)}_{j+1,1} \right] -$$

$$\left(\frac{\lambda}{\lambda+\mu}\right) \sum_{n=0}^{\infty} \left\{ \left[ e^{-(\lambda+\mu)t} \cdot \sum_{r=1}^{n+1} \frac{[t(\lambda+\mu)]^{n-r+1}}{(n-r+1)!} \right] \cdot \left( P^{*(n)}_{j,1} - P^{*(n)}_{j+1,1} \right) \right\} +$$

$$e^{-(\lambda+\mu)t} \sum_{k=1}^{i} \left[ \sum_{n=0}^{\infty} \frac{[t(\lambda+\mu)]^n}{n!} \left( P^{*(n)}_{j,k} - P^{*(n)}_{j+1,k} \right) \right] \qquad (14)$$

$P_{i,0}(t) =$

$$1-\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty} P_{1,1}^{*(n)}+\left(\frac{\lambda}{\lambda+\mu}\right)\sum_{n=0}^{\infty} e^{-(\lambda+\mu)t} \cdot \sum_{r=1}^{n+1} \frac{[t(\lambda+\mu)]^{n-r+1}}{(n-r+1)!} \cdot P_{1,1}^{*(n)}$$

$$-e^{-(\lambda+\mu)t} \sum_{k=1}^{i}\left[\sum_{n=0}^{\infty} \frac{[t(\lambda+\mu)]^n}{n!} P_{1,k}^{*(n)}\right] \qquad (15)$$

Use $\sum_{n=0}^{\infty} P_{j,k}^{*(n)} = \delta_{jk} + \frac{\rho_{jk}}{1-\rho_{kk}} < \infty .$

Then for j>0, $\pi_j = \frac{\lambda}{\lambda+\mu}\left(\delta_{j1}-\delta_{j+1,1}+\frac{\rho_{j1}-\rho_{j+1,1}}{1-\rho_{11}}\right)$ if it exits

and $\pi_0 = 1-\frac{\lambda}{\lambda+\mu}\left(1+\frac{\rho_{11}}{1-\rho_{11}}\right)$ if it exists. (16)

It is known that $\pi_j = (1-\rho)\rho^j$ for some ρ determined numerically by root finding technique. We provide a combinatorial expression for ρ.

$$\rho = \frac{\lambda}{\lambda+\mu}\left(1+\frac{\rho_{11}}{1-\rho_{11}}\right).$$

$\rho_{j1}$ represents the probability of going from j and eventually reaching at 1 (i.e., not reaching earlier) which is to happen in n steps, n= 1,2,…We use the same counting argument as before.

Then,

$$\rho_{j1} = \sum_{n=1}^{\infty} \frac{j-1}{n} \binom{n}{\frac{n-j+1}{H+1}} \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{n-j+1}{H+1}} \left(\frac{\lambda}{\lambda+\mu}\right)^{\frac{Hn+j-1}{H+1}}$$

$$\rho_{11} = \sum_{n=2}^{\infty} \frac{H-1}{n-1} \binom{n-1}{\frac{n-H}{H+1}} \left(\frac{\lambda}{\lambda+\mu}\right)^{\frac{n}{H+1}} \left(\frac{\mu}{\lambda+\mu}\right)^{\frac{Hn}{H+1}}$$

.

It provides an alternative combinatorial expression for $\pi_j$.

Concluding Remark:

We can deal with queues having batches of different sizes and with catastrophes (i.e., there is a constant probability of reaching 0 from any state).

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