GARCH Option Pricing: a Semiparametric Approach

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Abstract

Option pricing based on GARCH models is typically obtained under the assumption that the random innovations are standard normal (normal GARCH models). However, these models fail to capture the skewness and the leptokurtosis in financial data. We propose a new method to compute option prices using a non-parametric density estimator for the distribution of the driving noise. We investigate the pricing performances of this approach using two different risk neutral measures: the Esscher transform pioneered by Gerber and Shiu (1994), and the extended Girsanov principle introduced by Elliot and Madan (1998). Both measures are justified by economic arguments and are consistent with Duan's (1995) local risk neutral valuation relationship (LRNVR) for normal GARCH models. The main advantage of the two measures is that one can price derivatives using skewed or heavier tailed innovations distributions to model the returns. An empirical study regarding European Call option valuation on S&P500 Index shows (i) under both risk neutral measures our semiparametric algorithm performs better than the existing normal GARCH models if we allow for a leverage effect and (ii) the pricing errors when using the Esscher transform are quite small even though our estimation procedure is based only on historical return data.

Keywords: Option pricing, GARCH, extended Girsanov principle, Esscher transform, kernel density estimator, semiparametric pricing

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1 Introduction

The theory of option pricing has been one of the major areas of interest in the financial literature. The seminal works of Black and Scholes (1973) and Merton (1973) were the starting point in deriving a closed form solution for European option prices, assuming the underlying asset price follows a one-dimensional log-normal diffusion process. In the Black-Scholes model the volatility has been considered constant, while in the Merton model the volatility is a deterministic function of time. The assumptions made in these models have been widely criticized, since many empirical studies have shown strong evidence that the implied volatility surface displays smile or smirk (skew) effects. There are several different explanations for why volatility exhibits skewness. For example, in most markets, the returns tend to leptokurtic than the normal distribution as assumed in the Black-Scholes model. Therefore a large number of papers proposed models which incorporate a stochastic volatility. There are two important directions in the literature regarding these type of models: continuous-time stochastic volatility processes represented in general by a bivariate diffusion process, and the discrete time autoregressive conditionally heteroscedastic (ARCH) model of Engle (1982) or its generalization (GARCH) as first defined by Bollerslev (1986). The most significant contribution in the continuous-time class of models incudes the works of Wiggins (1987), Hull and White (1987), Scott (1987), Stein and Stein (1991) and Heston (1993). Another innovative idea has been to include jumps in either the asset price process as done in Bakshi, Cao and Chen (1997) and Scott (1997) or in both asset price and volatility specification as proposed by Duffie et al. (2000). More recently, Carr, Geman, Madan and Yor (2003) investigate the option pricing performance of a time-changed Levy process. Even if the continuous time models presents advantages in constructing closed form solutions for European option prices, their Markovian structure is not consistent with the empirical findings. Thus, in the last few years, more interest has been given to the discrete-time GARCH option pricing models. The most important papers which study the empirical fitting of these models includes Pagan and Schwert (1990), Glosten et al. (1993), and Bollerslev et al. (1994). Another advantage of GARCH models is that the volatility is observable at each time point, thus making the estimation procedure a much easier task than the one in continuous-time models, as well as yielding some diagnostic tools. In the GARCH models, option prices are evaluated as discounted expected value of the payoff function under a martingale measure. It is well known that in the GARCH setup the markets are incomplete, so there exists contingent claims which cannot be replicated exactly by constructing a self-financing hedge. Therefore there is an infinite number
of risk neutral measures under which one can price derivatives. In general, the possible choices of such measures have to be justified by economic foundations. For incomplete models one may wish to find a self financing strategy which is the best with respect to some optimality criteria. Schweizer (1988) introduced the notion of local risk minimization which involves finding strategies that minimize the one step ahead expected quadratic cost process. Follmer and Schweizer (1991) showed that the minimal martingale measure is consistent with this criteria. Elliot and Madan (1998) use an extended Girsanov principle to construct a risk neutral measure which is supported by finding similar hedging strategies. The minimal entropy martingale measure was also studied in the financial literature. Fritelli (2000) gave a general characterization of the density of the minimal entropy martingale measure in discrete time. Connections between these various measures have been done using the so-called p-optimal measure which minimizes the p-th moment of the Radon-Nycodim derivative of the risk neutral measure with respect to the underlying measure; the minimal entropy measure and the variance-optimal measure can be viewed as special cases $p \to 1$ and $p = 2$ (see for example Grandits (1999) or Delbaen et al. (1997) ). Another well-known tool widely used in actuarial science is the Esscher transform. Gerber and Shiu (1994a) were the first to apply this principle to option pricing. They showed that the choice of the Esscher transform is justified by maximizing the expected power utility function of a representative economic agent. Chan (1999) showed the relationship between the minimal entropy martingale measure and the Esscher transforms when the stock price is driven by a Levy process.

Option pricing in GARCH models has been typically done using the local risk neutral valuation relationship (LRNVR) pioneered by Duan (1995). He justified the choice of this measure by the argument that the representative agent in an economy is an expected utility maximizer, and the utility is additive and time-separable. The crucial assumptions in his construction are the conditional normality distribution of the asset returns under the underlying probability space and the invariance of the conditional volatility to the change of measure. The empirical performance of these normal GARCH option pricing models has been studied by many authors, for example in Amin and Ng (1993), Duan (1996), and Hardle and Hafner (2000). In a more recent study Christoffersen and Jacobs (2004) investigate the pricing performances of GARCH models with different volatility specifications. They also compare the errors in option pricing by these models when the GARCH parameters are estimated in two different ways: by QMLE using only information about the historical returns and by calibration procedure which consists
in minimizing the mean squared errors between the observed market option prices and the model
prices. In all these papers option prices were computed using Monte-Carlo simulation. How-
ever, Heston and Nandi (2000), derived a semi-analytical pricing formula for European options
for a leverage normal GARCH model. The advantage of their closed-form solution is that the
calibration technique is much easier to implement.

In general the skewness and leptokurtosis of the financial data cannot be captured by a
GARCH model with innovations which are normally distributed. Bai et al. (2003) showed
that the returns kurtosis implied by a normal GARCH model is typically much smaller than
the sample kurtosis observed for most returns time series. There are only a few papers in the
literature which investigate the option pricing for GARCH models with non-normal innovations.
Siu et al. (2004) propose a GARCH option pricing model when the driving noise is Gamma
distributed and based on the Esscher transform. Following the line of Heston and Nandi (2000),
Christoffersen et al. (2004) derived a closed-form solution for a GARCH option pricing model
with Inverse Gaussian innovations. Menn and Rachev (2005) studied the pricing performance of
a smoothly-truncated stable GARCH model. Duan (1999) proposes a new generalized LRNVR
for GARCH models when the returns are conditionally skewed and fat-tailed distributed. This
approach was also applied by Stentoft (2005) for pricing American options under GARCH mod-
els and by Christoffersen et al. (2006b) for pricing European options under a two volatility
component GARCH models with GED innovations. A new class of GARCH models was intro-
duced by Duan et al. (2006) by allowing both return and volatility dynamics to include jumps.
Their option pricing theory is done in a similar way as in Duan (1995). Another important
direction in the financial literature is to estimate the risk-neutral return distribution and risk-
neutral return volatility dependence using non-parametric techniques as in Ait-Sahalia and Lo
(2000a, 2000b).

In this paper we propose a new semiparametric algorithm to derive European option prices
under a general GARCH framework. Various indices and stocks may behave differently, so
assuming a particular parametric family of distributions for the driving noise may not be ap-
propriate. To overcome this problem we propose a GARCH model for which we do not require
any distributional assumption for the innovations. A related idea was used by Barone-Adesi
et al. (2004) for pricing and hedging under GARCH incomplete models. Their methodology
consists of two steps: first, they estimate the GARCH parameters and scaled residuals using the
historical returns; in the second step, they calibrate the model to the observed market prices by
simulating risk-neutralized stock prices using the standardized residuals, instead of the Gaussian innovations. However, their calibration procedure is somehow artificial in the sense that they use the same risk-neutralized conditional mean process as in the normal case.

In our approach we estimate the GARCH parameters by the QMLE technique and then by approximating the unknown innovation distribution function using a kernel density estimator based on the standardized residuals. The kernel density estimator is in general mixture density, therefore deriving a risk-neutralized return dynamic in this case may be very tedious depending on the choice of risk neutral measure.* In order to avoid these problems, we propose an algorithm to compute option prices by simulating stock prices under the underlying historical measure and by evaluating Radon-Nycodim derivatives in accordance with the Esscher transform and the extended Girsanov principle. The form of the derivative is dictated by the choice of the kernel function.

We implement this algorithm, using observed S&P500 Index data, to price European Call options on this index. In our simulation study, we consider a simple GARCH specification for the returns and the threshold GARCH (TGARCH) specification, which allows for leverage effects. We compare the pricing performance of our methodology to the GARCH/TGARCH models driven by Gaussian and shifted Gamma innovations. Using two option data sets we have found the semiparametric TGARCH option pricing model evaluated using both risk neutral measures is superior to both the normal and shifted Gamma TGARCH. It is worth remarking that the implementation of the Esscher transform in the semiparametric framework is very promising. It appears to provide an overall the best pricing strategy, despite the fact that it uses only observed returns. It prices quite well both in-the-money and out-of money long term options. This paper does not evaluate the performance of semiparametric GARCH models calibrated to observed option prices. An interesting future research topic is to compare in-the-sample and out-of-sample pricing errors of these models versus the continuous-time stochastic volatilities candidates.

The remainder of this paper is organized as follows. In Section 2 we describe the general GARCH model; In Subsections 2.2 and 2.3 we review briefly two martingale measures in a general discrete time setup: the Esscher transform and the Elliot and Madan’s Extended Girsanov principle. We also show that these measures are consistent with Duan’s LRNVR for GARCH.

*The risk neutral valuation relation can also be extended for general mixture models; for a detailed discussion we refer to Garcia et al. (2003).
models driven by normal innovations and that they also solve a minimal entropy martingale problem. Section 3 is dedicated to the construction of the semiparametric pricing algorithm. The validity of this methodology is tested in a numerical experiment in Section 4. Another interesting problem is the sensitivity of the pricing algorithm to the choice of the kernel and the bandwidth in our nonparametric density estimator. Section 5 undertakes a numerical experiment to discuss this question. In Section 6 we present the empirical performance of these semiparametric models using two option data sets. The conclusions and related future research is presented in the Section 7.

2 Risk neutral measures under GARCH models

2.1 Definitions and notations

Consider a discrete time economy with the time index \( T = \{t|t = 0, 1, ...T\} \) of trading dates consisting of one risk-free asset and one risky stock. Let \((\Omega, \mathcal{F}, \mathcal{F}_{t}, P)\) be a complete filtered probability space, where \(P\) is the historical probability measure and \(\mathcal{F}_{t}\) is a sequence of increasing \(\sigma\)-fields of \(\mathcal{F}\) representing all market information up to time \(t\); we assume \(\mathcal{F}_0 = \{0, \Omega\}\) and \(\mathcal{F}_T = \mathcal{F}\).

Let \((S_0^0, S) = (S_0^0, S_t)_{0 \leq t \leq T}\) be the price process for the reference asset and the risky stock. In our discrete time setting we assume the stock price process \(S_t\) is adapted to the filtration \(\mathcal{F}\) and the risk-free asset price process \(S_0^0\) is a deterministic process with the dynamic \(S_0^0 = e^{-rt}\), where \(r\) represents the continuously compounded one-period risk free interest rate. The discounted stock price is thus \(\tilde{S}_t = e^{-rt}S_t\). We also denote by \(y_t = \ln \frac{S_t}{S_{t-1}}\) the continuously compounded return process with the dynamic given by the following GARCH structure:

\[
\begin{align*}
y_t &= m_t + \sqrt{h_t}\varepsilon_t \quad (1) \\
h_t &= \alpha_0 + \alpha_1 h_{t-1}\omega(\varepsilon_{t-1}) + \beta_1 h_{t-1} \quad (2)
\end{align*}
\]

the driving noise random variables \(\varepsilon_t\) are iid with mean zero and unit variance and \(m_t\) is a predictable process which may depend on the conditional variance \(h_t\) and some other unknown parameters. We denote with \(\theta\) the vector of all unknown parameters, which are to be estimated by appropriate observed data, from the model. The choice of \(\omega\) dictates the type of GARCH model one wants to analyze; depending on its form and on the distribution of the innovations we have to impose the usual constraints on the GARCH parameters to ensure the positivity of
$h_t$ and the covariance stationarity of the model. For example, in the simple GARCH(1,1) model
\[ \omega(\varepsilon_{t-1}) = \varepsilon_{t-1}^2. \]

The filtration $\mathcal{F}_t$ is the usual filtration generated by the GARCH innovations, $\mathcal{F}_t = \sigma(\varepsilon_u, u \leq t)$. In general the information available to an agent is the $\sigma$-field $\mathcal{G}_t = \sigma(y_u, u \leq t)$ which contains the information only about the evolution of the prices, that is the weak form of market efficiency. However, assuming $h_0$ is known in our GARCH setup, then $\sqrt{h_t}$ is $\mathcal{G}_{t-1}$ measurable, so $\varepsilon_t$ is $\mathcal{G}_t$ measurable and $\mathcal{F}_t = \mathcal{G}_t$. This relation is not true for all discrete time models.

To avoid arbitrage opportunities, we assume that the price process admits an equivalent martingale measure. A probability measure $Q$ is an equivalent martingale measure w.r.t. to $P$ if:

- $Q \approx P$ (i.e. $\forall B \in \mathcal{F}, Q(B) = 0 \iff P(B) = 0$)
- the discounted price process $\tilde{S}_t$ is a martingale under $Q$ w.r.t. to $\mathcal{F}_t$, that is $E^Q[\tilde{S}_t|\mathcal{F}_{t-1}] = \tilde{S}_{t-1}$.

We denote with $Q^e(P)$ the set of all martingale measures equivalent with $P$ (risk neutral measures). It is well known that if prices are modeled by GARCH processes, the market is incomplete; therefore there may be an infinite number of risk neutral measures to price contingent claims.

There are many papers in the financial literature dealing with possible choices of a risk neutral measure. For the GARCH option pricing models, most of the studies use the local risk neutral valuation principle (LRNVR) introduced by Duan (1995).

A risk neutral measure $Q \in Q^e(P)$ satisfies the LRNVR if:

- the returns $y_t$ are conditionally normally distributed under $Q$
- $\text{Var}^P[y_t|\mathcal{F}_{t-1}] = \text{Var}^Q[y_t|\mathcal{F}_{t-1}]$.

This choice of $Q$ can be justified by the argument that the representative agent in an economy is an expected utility maximizer for constant relative risk aversion utilities and the relative changes in aggregate consumption are normally distributed. Another nice aspect is that we can easily obtain the dynamic of the return process under $Q$:

\[ y_t = r - \frac{1}{2}h_t + \sqrt{h_t} \eta_t \]  
\[ h_t = \alpha_0 + \alpha_1 h_{t-1} \omega \left( \frac{r - h_{t-1}/2 - m_{t-1}}{\sqrt{h_{t-1}}} + \eta_{t-1} \right) + \beta_1 h_{t-1} \]
where $\eta_t \sim N(0, 1)$ and independent under $Q$. The risk neutralized dynamic of the returns allows us to compute option prices by simulating the price process under $Q$. Unfortunately the normality assumption of the driving noise is too restrictive and, as shown in numerous empirical studies, it cannot capture the skewness and the kurtosis of the financial data. In the past few years a small number of researchers have investigated the performance of the GARCH option pricing models based on some other distributional assumption for the innovations. For example, Christoffersen et al (2004) proposed a GARCH model with Inverse Gaussian driving noises, Menn and Rachev (2005) used a smoothly truncated stable distribution, while Duan et al (2006) studied GARCH models with jumps.

2.2 The Conditional Esscher Transform

The conditional Esscher transform for a general discrete setting was introduced by Buhlmann et al. (1996) [10] as a generalization of the well-known Gerber-Shiu’s (1994a) Esscher transform.

It is assumed that the conditional moment generating function of $y_t$ w.r.t. $\mathcal{F}_{t-1}$ exists for all $t$, $0 \leq t \leq T$:

$$M^P_{y_t|\mathcal{F}_{t-1}}(c) = E^P[e^{cy_t}|\mathcal{F}_{t-1}] < \infty, \ c \in D \subseteq \mathbb{R}.$$  

**Definition 2.2.1** Let the process $Z_t$ defined by:

$$Z_t = \prod_{k=1}^{t} e^{\frac{\delta^*_k y_k}{M^P_{y_k|\mathcal{F}_{k-1}}(\delta^*_k)}}$$  

where $Z_0 = 1$ and $\delta^*_k$ is a predictable process and $\delta^*_k$ is the unique solution of the equation:

$$M^P_{y_k|\mathcal{F}_{k-1}}(1 + \delta_k) = e^{\delta_k} M^P_{y_k|\mathcal{F}_{k-1}}(\delta_k)$$  

for all $k \in \Gamma \ldots T$. The conditional Esscher transform $Q^{\text{ess}}$ w.r.t. $P$ of the process $y_t$ is defined as:

$$\frac{dQ^{\text{ess}}}{dP} = Z_T$$  

It is straightforward to show that the process $Z_t$ is a $P$-martingale. The martingale property of the discounted price process $\tilde{S}_t$ under the Esscher measure is guaranteed by equation (6).

There are various justifications in the literature for the choice of the Esscher transform in derivatives valuation. One approach is presented in Gerber and Shiu (1994b) where they showed
that the Esscher change of measure is a natural choice if the representative agent is an expected utility maximizer for power utility functions. A second direction mentioned in the literature is that the Esscher transform can be obtain by solving a minimal entropy problem subject to some specific constraints, usually moment constraints.

The advantage of this measure over Duan’s LRNVR is that it can be applied in the GARCH framework for any type of distribution as long as its moment generating function exists. Tak Kuen Siu et al. (2004) construct European option prices based on the Esscher transform assuming a gamma distribution for the driving noise. Christoffersen et al. (2004) use it for the Inverse Gaussian case. Proposition 2.2.1 gives the relation between the Esscher transform and LRNVR when the returns follow a normal GARCH.

**Proposition 2.2.1** Suppose that the dynamic of \( y_t \) is given by equations (1) and (2) under \( P \) and let \( \varepsilon_t \sim N(0, 1) \). Then the Esscher transform and the LRNVR are the same.

**Proof** The conditional moment generating function of \( y_t \) under \( Q^{ess} \) is

\[
M_{y_t|\mathcal{F}_{t-1}}^{Q^{ess}}(c) = E^Q[\exp(c y_t)|\mathcal{F}_{t-1}] = \frac{M_{y_t|\mathcal{F}_{t-1}}^P(c + \delta^*_t)}{M_{y_t|\mathcal{F}_{t-1}}^P(\delta^*_t)} = \exp\left(c(m_t + \delta^*_t h_t) + c^2 h_t^2 / 2\right)
\]

which is the mgf on a normal distribution with mean and variance

\[
E^{Q^{ess}}[y_t|\mathcal{F}_{t-1}] = m_t + \delta^*_t h_t
\]

\[
Var^{Q^{ess}}[y_t|\mathcal{F}_{t-1}] = h_t^2.
\]

The solution of the martingale condition from equation 6 is:

\[
\delta^*_t = \frac{1}{h_t}(r - m_t - h_t / 2)
\]

Plugging \( \delta^*_t \) in the above expectation we get:

\[
E^Q[y_t|\mathcal{F}_{t-1}] = r - \frac{1}{2} h_t
\]

Thus we can conclude that under \( Q^{ess} \) the Esscher method and the local-RNVR leads to the same risk-neutralized dynamic of the returns given by (3) and (4).

This result was also mentioned in Siu et al (2004) for two particular choices of \( m_t \). We argue here that the result holds for any form of the mean of \( y_t \). The form of \( m_t \) will only affect the variance specification.

The link between the Esscher transform and the minimal entropy martingale measure for the GARCH models is given in Proposition 2.2.2.
**Definition 2.2.2** Let $P$ and $P'$ be two equivalent probability measures. The relative entropy $I(P', P)$ is defined by:

$$I(P', P) = E^{P'} \left[ \ln \frac{dP'}{dP} \right]$$

In our GARCH setting we define the minimal entropy martingale measure (MEMM) by limiting our searching procedure on a smaller class of martingale measures; more precisely we investigate only the class of transformations which preserve the distribution of the returns prior to the change of measure; moreover, using a Girsanov type argument from the diffusion process theory, we assume that the returns volatility remains unchanged in the risk neutral setup.

**Definition 2.2.3** The MEMM $Q^{MEMM}$ is the solution of the following minimum problem:

$$\min_{P'} I(P', P)$$

subject to the constraints:

$$E^{P}[\frac{dP'}{dP}] = 1$$

$$E^{P'}[\frac{S_t}{S_{t-1}}|\mathcal{F}_{t-1}] = e^{r}.$$  

**Proposition 2.2.2** For the GARCH model with normal innovations $Q^{ess}$ is the unique solution for the minimization problem given in Definition 2.2.3.

**Proof** The proof consists in solving a simple optimization problem using the standard procedure. We also have to remark that in the normal case the second constraint ensure that the mean under the new measure is $r - \frac{1}{2} h_t$.

Another noteworthy aspect is the relation between the Esscher transform and choosing a particular form for the pricing kernel. In the next proposition we show that the change of measure induced by a log-linear pricing kernel is consistent with an Esscher transform, and this is valid regardless any assumption one would want to make about the bivariate conditional distribution of the returns and the logarithm of the stochastic discount factor (SDF).

**Proposition 2.2.3** Let $M_t$ be the pricing kernel defined as the ratio of the marginal utility of consumption at time $t$ and $t-1$, and assume that the following equilibrium conditions are satisfied:

A1. $E^P[M_t|\mathcal{F}_{t-1}] = e^{-r_t}$
A2. $E^P[M_t e^{y_t} | \mathcal{F}_{t-1}] = 1$.

Let $Q^M$ be defined by its Radon-Nycodim derivative:

$$\frac{dQ^M}{dP} := Z_T = e^{rT} \prod_{t=1}^T M_t$$

(8)

with $Z_0 = 1$. If the SDF has the following representation:

$$\log M_t = \delta_1 y_t + \delta_2$$

(9)

for some predictable processes $\delta_1$ and $\delta_2$, then the probability measure $Q^M$ defined by (8) is an Esscher transform with respect to the return process $y_t$ and Esscher parameter $\delta_1$.

**Proof** The fact that $Q^M$ is a risk neutral probability measure equivalent to $P$ follows immediately applying Bayes rule (see Lemma 1, page 6, Duan et al. (2005)). We need only to show that $Q^M$ and $Q^{ess}$ are the same. Plugging in (9) into (8) we get:

$$\frac{dQ^M}{dP} = e^{rT} \prod_{t=1}^T e^{\delta_1 y_t + \delta_2}.$$ 

(10)

It remains to find the predictable processes $\delta_1$ and $\delta_2$ which satisfies the no-arbitrage conditions A1 and A2. From assumption A2 we get that:

$$M^P_{y_t | \mathcal{F}_{t-1}} (1 + \delta_1) = e^{-\delta_2}$$

and then from A1 we have that:

$$M^P_{y_t | \mathcal{F}_{t-1}} (1 + \delta_1) = e^r M^P_{y_t | \mathcal{F}_{t-1}} (\delta_1^*).$$

Since the solution of this equation is unique we have $\delta_1 = \delta_1^*$ from the Esscher transform. Replacing these in (10) we get:

$$\frac{dQ^M}{dP} = \prod_{t=1}^T \frac{e^{\delta_1 y_t} e^{r}}{M^P_{y_t | \mathcal{F}_{t-1}} (1 + \delta_1)} = \prod_{t=1}^T \frac{e^{\delta_1^* y_t}}{M^P_{y_t | \mathcal{F}_{t-1}} (\delta_1^*)} = \frac{dQ^{ess}}{dP}$$

which completes our proof.

Thus we have provide here another natural explanation of why one would choose the Esscher transform in the sense that it is justified by a log-linear pricing kernel as a function on $y_t$. 
2.3 Extended Girsanov Principle

The method introduced by Elliot and Madan (1998) represents another natural alternative in choosing a risk neutral measure in a discrete time setting. The motivation for their transformation is based on their extended Girsanov principle, that is, the discounted stock price, under the new measure, has to follow the law of their martingale component prior to the change of measure. Their transformation comes also as a correction to the minimal martingale measure of Follmer and Schweizer (1991) in the sense that the latter yields a signed measure, which is not always a probability measure in discrete time market models. The Elliot and Madan (1998) method is only weak form efficient in the sense that the filtration associated to the complete probability space will contain information only about the evolution of the stock price. However, we showed earlier that in a GARCH setup these two filtration coincide. The construction of the new measure relies on the multiplicative decomposition of the discounted stock price.

In order to have such a representation they assume that $E^P[\tilde{S}_t] < \infty$. We recall $\mathcal{G}_t$ is the sequence of $\sigma$-algebras generated by the returns, $\mathcal{G}_t = \sigma(y_u, u \leq t)$. Recall in the GARCH setting $\mathcal{G}_t = \mathcal{F}_t$. For completeness we now describe the multiplicative decomposition of Elliot and Madan.

\begin{equation}
\tilde{S}_t = \tilde{S}_0 A_t M_t \tag{11}
\end{equation}

where $A_t$ is a predictable process w.r.t. $\mathcal{G}_t$ and $M_t$ is an $\mathcal{G}_t$ martingale. The process $A_t$ has the unique representation:

$$A_t = \prod_{k=1}^{t} E^P\left[ \frac{\tilde{S}_k}{\tilde{S}_{k-1}} | \mathcal{G}_{k-1} \right]$$

and where $M_t$ is defined by:

$$M_t = \frac{\tilde{S}_t}{\tilde{S}_0 A_t}.$$

The dynamic of the discounted stock price under $P$ in the following form:

\begin{equation}
\tilde{S}_t = \tilde{S}_{t-1} e^{\mu t} W_t \tag{12}
\end{equation}

where $W_t := M_t/M_{t-1}$ is a mean one $\mathcal{G}_t$-martingale under $P$ and $e^{\mu t}$ represents the one period discounted excess returns:

$$e^{\mu t} = e^{-r} E^P[e^{y_t} | \mathcal{G}_{t-1}].$$
**Definition 2.3.1** Let the process \( Z_t \) defined by:

\[
Z_t = \prod_{k=1}^{t} \frac{g_k^P\left(\frac{\tilde{S}_t}{S_{t-1}}\right)e^{\mu t}}{g_k^P(e^{-\mu t}\frac{\tilde{S}_t}{S_{t-1}})}
\]

where \( g_k^P(w_t) \) is the conditional pdf of \( W_t \) given \( G_{t-1} \). The risk neutral measure given by the extended Girsanov principle \( Q^{egp} \) is defined by:

\[
\frac{dQ^{egp}}{dP} = Z_T.
\]

Elliot and Madan showed that \( Q^{egp} \) is the unique probability measure such that

\[
\mathcal{L}^{Q^{egp}}\left(\frac{\tilde{S}_t}{S_{t-1}}\right) = \mathcal{L}^P(W_t).
\]

In particular \( \tilde{S}_t \) is a \( G_t \) martingale under \( Q^{egp} \).

As is done for the minimal martingale measure, the extended Girsanov principle can be justified by a hedging criteria. The choice of the extended Girsanov principle measure is supported by weak efficient hedging strategies that minimizes the conditional variance of the discounted risk-adjusted costs of hedging; see Theorems 4.1 and 4.2 of Elliot and Madan (1998).

The relation between the extended Girsanov principle and Duan’s LRNVR for the normal GARCH case is given in the following proposition.

**Proposition 2.3.1** The extended Girsanov principle provide the same risk neutral measure as the LRNVR when GARCH innovations are normally distributed.

**Proof** The proof is done in a similar way as for the Esscher transform, by evaluating the conditional moment generating function of the returns under \( Q^{egp} \).

To simplify the form of the Radon-Nycodim derivative in we remark that \( \{y_t\} \) has the following additive decomposition:

\[
y_t = r + \mu t + \ln W_t.
\]

Using a simple transformation we rewrite (14) in terms of the conditional density function of the returns

\[
\frac{dQ^{egp}}{dP} = \prod_{t=1}^{T} \frac{f_t^P(y_t - r + \ln M_{yt|\mathcal{F}_{t-1}})}{f_t^P(y_t)}
\]

where \( f_t^P(y_t) \) is the conditional pdf of \( y_t \) given \( G_{t-1} \). Moreover, since \( y_t|\mathcal{F}_{t-1} \sim \mathcal{N}(m_t, h_t) \) under \( P \), we have

\[
\frac{dQ^{egp}}{dP} = \prod_{t=1}^{T} \exp\left(y_tC_1(t) + C_2(t)\right)
\]
where \( C_1(t) \) and \( C_2(t) \) are two predictable processes w.r.t. \( \mathcal{F}_t \) and are given by

\[
C_1(t) = -\frac{mt + h_t/2 - r}{h_t} \\
C_2(t) = -\frac{1}{2h_t} \left( \frac{mt + h_t/2 - r}{h_t/2 - r} \right).
\]

Using the Bayes rule we compute the conditional moment generating function of \( Y_t \) given \( \mathcal{F}_{t-1} \) under \( Q^{egp} \).

\[
M_{Y_t|\mathcal{F}_{t-1}}^{Q^{egp}}(c) = E^{Q^{egp}}[\exp(cY_t)|\mathcal{F}_{t-1}] = E^P[\exp(cY_t)\frac{dQ^{egp}}{dP}|\mathcal{F}_{t-1}]
\]

\[
= \exp\left(C_2(t)\right)E^P[\exp\left(Y_t(c + C_1(t))\right)|\mathcal{F}_{t-1}]
\]

\[
= \exp\left[C_2(t) + (c + C_1(t))mt + (c + C_1(t))^2\frac{h_t}{2}\right]
\]

\[
= \exp\left(C_2(t)\right)\exp\left(C_1(t)mt + C_1^2(t)\frac{h_t}{2}\right)\exp\left((mt + C_1(t)h_t)c + c^2\frac{h_t}{2}\right).
\]

After imposing the condition for \( M_{Y_t|\mathcal{F}_{t-1}}^{Q^{egp}}(c) \) to be a moment generating function, that is \( M_{Y_t|\mathcal{F}_{t-1}}^{Q^{egp}}(0) = 1 \), we will immediately find that \( Y_t \) follows under \( Q^{egp} \) a normal distribution with conditional mean and variance given by

\[
E^{Q^{egp}}[Y_t|\mathcal{F}_{t-1}] = mt + C_1(t)h_t,
\]

\[
Var^{Q^{egp}}[Y_t|\mathcal{F}_{t-1}] = h_t.
\]

Substituting \( C_1(t) \) in the above mean equation we note that under \( Q^{egp} \) the return process \( Y_t \) and the volatility process \( h_t \) satisfy equations (3) and (4).

The consistency between Duan’s LRNVR and the conditional Esscher transform and the Extended Girsanov principle in the normal GARCH case are not surprising and both latter methods can be viewed as applications of the LRNVR in the conditional normality case. However, their main advantage is that one can further investigate the pricing performances using skewed and heavier tailed distributions to model the returns.

### 3 The semiparametric option pricing model

The goal of this study is to evaluate European options for the general GARCH model given by equations (1) and (2) by approximating the unknown true distribution of the returns under \( P \) with a nonparametric density estimator.
Recall that the price of an European Call option for a given equivalent measure $Q \in \mathcal{Q}^e(P)$ with strike $K$, maturity $T$ and payoff $h(S_T) = (S_T - K, 0)^+$ is given by $e^{-r(T-t)} E^Q[h(S_T)|\mathcal{F}_t]$. In general, in the GARCH framework there is no closed form for option pricing, so the above expectation is usually evaluated using a Monte Carlo technique. However, Heston and Nandi (2000) and more recently Christoffersen et al (2004) derived semi-analytic solutions for European option prices based on Gaussian, and Inverse Gaussian return distributions respectively.

When using nonparametric density estimators, the risk-neutralized dynamic of the returns is not a simple structure under either risk neutral measure described Section 2. Therefore our simulation study is based on the following Monte Carlo estimator:

$$\frac{1}{M} \sum_{m=1}^{M} h(S_T(m)) \frac{dQ}{dP}(m) \xrightarrow{M \to \infty} E^Q[h(S_T)]$$

(15)

where $S_T(m)$ are the $m$-th stock paths simulated under the physical measure $P$ and $\frac{dQ}{dP}(m)$ is the $m$-th path of the Radon-Nycodim derivatives computed based on them. The speed of convergence of the estimator from (15) and the usual Monte Carlo estimator constructed in the risk neutral world is measured in terms of the variances. When the returns distribution is also known under $Q$ one may want to test which Monte Carlo simulation is more efficient. In a simulation study, not reported here, we computed option prices using both estimators, for the former using either Esscher transform or Extended Girsanov principle and for the latter using equations (3) and (4). When the returns are conditionally normal distributed we found that simulation under physical measure $P$ is more efficient than the one using the risk neutralized dynamic of the returns. An intuitive explanation for this could be seen by examining the Esscher set of parameters $\delta_t^*$. For example if we take the conditional mean returns under $P$ to be $m_t = r + \lambda \sqrt{\theta_t}$, then it follows that $\delta_t^* = \frac{1}{\theta_t}(-\lambda \sqrt{\theta_t} - \frac{\theta_t}{2})$, where $\lambda$ is the positive risk premium parameter. We conclude that $\delta_t^* < 0$, $t = 0...T$ and thus the Esscher transform tilts the conditional returns density to the left. In the case of out-of money options one would expect to obtain less zero realizations of the payoffs when simulating stock price paths under the physical measure $P$ then under $Q^{ess}$, and this will lead to a variance reduction of the Monte-Carlo estimator of the form given by (15).

In the following, we propose a semiparametric Monte Carlo option pricing algorithm for European Call options when $y_t$ satisfies equation (1) and (2). The method is based on parametric estimates of the GARCH parameters and a nonparametric estimator for the unknown density of the driving noise. The model parameter estimates are obtained by the quasi-maximum likelihood technique (QMLE). For the nonparametric density estimator we use a kernel estimator based on the standardized residuals.
Algorithm 1

Step 1. Estimate \( \hat{\theta}_n \) by QMLE using all historical information \( \mathcal{F}_t \), where \( n \) is the number of observations up to time \( t \). Evaluate the standardized residuals \( \hat{\varepsilon} = (\hat{\varepsilon}_s)_{s \leq t} \):

\[ \hat{\varepsilon}_s = \frac{y_s - m_s(\hat{\theta}_n)}{\sqrt{h_s(\hat{\theta}_n)}}. \]

Step 2. For each \( m = 1 \ldots M \) generate randomly \( \hat{\varepsilon}^*(m) = (\hat{\varepsilon}^*_t(m), \ldots, \hat{\varepsilon}^*_T(m)) \) from the kernel density estimator of innovation distribution given by:

\[ \hat{p}_n(x, \hat{\varepsilon}_n) = \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{x - \hat{\varepsilon}_i}{h}\right) \] (16)

where \( h \) is the bandwidth and \( k(\cdot) \) is the kernel function.

Step 3. Simulate recursively the variance and return process using equations (1) and (2):

\[ h_{s+1}(\hat{\theta}_n) = \hat{\alpha}_0 n + \hat{\alpha}_1 n h_s(\hat{\theta}_n) + \hat{\beta}_1 n h_s(\hat{\theta}_n) \]
\[ y_{s+1}(\hat{\theta}_n) = m_{s+1}(\hat{\theta}_n) + \sqrt{h_{s+1}(\hat{\theta}_n)} \hat{\varepsilon}^*_{s+1} \]

where \( t \leq s < T \). The mth path of the simulated stock price is given by:

\[ S_T(m, \hat{\theta}_n) = S_t \sum_{s=t+1}^{T} e^{y_s(\hat{\theta}_n)}. \]

Step 4. The values of an European Call option of time \( t \) are given by:

\[ C_{t}^{\text{ess}}(\hat{\theta}_n) = \frac{1}{M} \sum_{m=1}^{M} e^{-r(T-t)} (S_T(m, \hat{\theta}_n) - K, 0) + \frac{dQ^{\text{ess}}}{dP}(m, \hat{\theta}_n) \] (17)
\[ C_{t}^{\text{egp}}(\hat{\theta}_n) = \frac{1}{M} \sum_{m=1}^{M} e^{-r(T-t)} (S_T(m, \hat{\theta}_n) - K, 0) + \frac{dQ^{\text{egp}}}{dP}(m, \hat{\theta}_n) \] (18)

where \( dQ^{\text{ess}}/dP \) and \( dQ^{\text{egp}}/dP \) are computed using the Esscher method and the Extended Girsanov principle respectively.

This procedure requires some further comments. First, it is well known that the GARCH parameters \( \hat{\theta}_n \) are estimated precisely only if we use a long time series for the past returns. Under some regularity assumptions we can argue the consistency of the QMLE estimator and the uniform consistency of the kernel density estimator based on the standardized residuals.
obtained at Step 1 are satisfied*, that is

$$\hat{\theta}_n \xrightarrow{a.s.} \theta_0$$

$$\sup_{x \in \mathbb{R}} \left| \hat{p}_n(x, \hat{\varepsilon}_n) - p_n(x, \hat{\varepsilon}) \right| = o_P(1)$$

where $\theta_0$ is the true model parameter and $p_n(x, \hat{\varepsilon})$ is the classical kernel density estimator based on the iid innovations $\varepsilon_1, ... \varepsilon_n$:

$$p_n(x, \hat{\varepsilon}) = \frac{1}{nh} \sum_{i=1}^{n} k \left( \frac{x - \varepsilon_i}{h} \right).$$

In simulating the volatility process at Step 3 we need to specify an initial value for the conditional variance. As stated in many other papers, a reliable choice is to use the last estimate of the conditional volatility.

In order to evaluate the Radon-Nycodim derivatives from Step 4 we need to choose the bandwidth specific to the kernel used in Step 2. There are many choices one can use for the kernel function, for example Gaussian, logistic, Epanechnikov etc. In our empirical study from Section 6 we use the Gaussian Kernel with the bandwidth chosen in order to minimize an $L^2$ distance between the density estimate and the true density as suggested in Silverman (1986):

$$k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

$$h_{L^2}^G = 1.06 \hat{\sigma} n^{-0.2}. \quad (20)$$

where $\hat{\sigma}$ is the empirical standard deviation of the variables $\varepsilon_t$ and the subscript $G$ denotes the Gaussian kernel. In Section 5 we numerically examine whether the choice of the kernel function and reasonable changes to the bandwidth have a major effect, and will argue the choice of kernel is does not have a significant influence on the results. Generating iid random variable from the Gaussian kernel density estimator is done using the following methodology:

1. Generate $Y$ from $\mathcal{N}(0,1)$.

2. Generate $U$ from $\mathcal{U}(0,1)$.

3. Let $\varepsilon^* = \sum_{i=1}^{n} (\hat{\varepsilon}_i + hY) I(\frac{i-1}{n} \leq U < \frac{i}{n}) = \hat{\varepsilon}_{[nU]+1} + hY$.

*the consistency and the asymptotic normality of these kernel density estimators based on standardized residuals for GARCH models have been recently discussed by a number of papers; in general the regularity conditions which have to be imposed are rather technical and it is not the objective of our paper to present them.
We need to make sure that, in simulating these random variables, standardized versions of \( (\epsilon_{t+1}(m), ..., \epsilon_T(m)) \) are generated at Step 2. This sampling procedure is fast and easy to implement. The only factor which is time consuming for our pricing algorithm is the computation of the Radon-Nynodim derivatives at Step 4. Pricing using the Extended Girsanov Principle is faster than the using the Esscher transform since for the latter we need to solve numerically for the Esscher parameter \( \delta_t^* \) for each \( t \in 0...T \).

4 Semiparametric Pricing Method: Numerical Experiment to Demonstrate Validity

In this section we examine the validity of our semiparametric option pricing method. In order to do this, we present a two stage numerical simulation experiment. First, we construct option prices in an ideal case where all model parameters are known and the “true” distribution of the GARCH innovations is given by a mixture of two normals (an \( MN_2 \)-distribution). In this setup, we evaluate the prices of European Call options for different strike prices using the extended Girsanov principle and the Esscher method. Throughout this section we refer to them as the “true” option prices. In the second part of this study, we apply the semiparametric algorithm from Section 3 to construct prices using nonparametric kernel density estimator to approximate the true innovation distribution, which is an \( MN_2 \).

For this purpose we recall the GARCH-in-mean specifications from equations (1) and (2) where \( m_t = r + \lambda \sqrt{h_t} \) and \( \epsilon_t \sim MN_2(0, 1) \). The parameter \( r \) represents the one period risk-free rate and \( \lambda \) is usually interpreted as the risk premium. For convenience we chose the simple GARCH specification for the conditional variance process by letting \( \omega(\epsilon_t) = \epsilon_t^2 \).

The choice of \( r \) and the GARCH parameter \( \theta_0 = (\alpha_0, \alpha_1, \beta_1, \lambda) \) can be made arbitrarily, making sure the usual constraints to ensure the positivity of \( h_t \) and covariance stationarity of the model are satisfied. However, in order for our results to be comparable with financial data, we set the true parameter \( \theta_0 \) to be the value as estimated by the QMLE method using the S&P 500 daily stock data. The estimates are reported in Table 1. We use daily closing prices of the S&P 500 index from January 04, 1988 to January 03, 2004 for a total of 4290 of observations. For the risk free rate \( r \), we assume a daily rate of \( 1.62 \times 10^{-4} \) which corresponds to an annual rate of 4.08\%, when taking an year to be 252 trading days. The persistence of the model implied by these parameters is 0.995 which is in accord with what usually is found in the financial literature.
Table 1: GARCH parameters estimators and their standard errors for S&P500 obtained by QMLE

<table>
<thead>
<tr>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\beta_1$</th>
<th>$\lambda_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5.096 \cdot 10^{-7}$</td>
<td>0.045</td>
<td>0.95</td>
<td>0.043</td>
</tr>
<tr>
<td>$(9.2 \cdot 10^{-8})$</td>
<td>(0.0034)</td>
<td>(0.0037)</td>
<td>(0.0153)</td>
</tr>
</tbody>
</table>

We consider the following simplest non-Gaussian form for the innovation distribution under $P$:

$$p(x) = \pi_1 \phi(x, a_1, b_1) + \pi_2 \phi(x, a_2, b_2)$$

(21)

with the normal mixture parameters arbitrarily chosen to satisfy the zero mean and the unit variance conditions; they are given in Table 2:

Table 2: Parameters of the MN2(0,1) distribution for $\epsilon_t \sim p$

<table>
<thead>
<tr>
<th>$\pi_1$</th>
<th>$\pi_2$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$b_1$</th>
<th>$b_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>0.25</td>
<td>-0.1</td>
<td>0.3</td>
<td>1.25</td>
<td>0.13</td>
</tr>
</tbody>
</table>

In the first step the “true” option prices are computed easily using a similar Monte Carlo simulation as the one from Algorithm 1.

**Stage 1**
- generate innovations $\varepsilon_{t+1}, \ldots, \varepsilon_T$ from the MN2(0,1) distribution given by (21);
- compute the “true” option prices $C_{0(2)}^{ess}(\theta_0)$ and $C_{0(2)}^{egp}(\theta_0)$ simulating the GARCH price paths based on the above driving noise and $\theta_0$.

The Radon-Nycodim derivatives need to be derived using the change of measures given by formula (7) and (14) for the mixture of two normal case. The Monte Carlo prices and their estimated variances are reported in the second and the third columns of Tables 3 and 4.

The goal of the second part of this experiment is to construct consistent estimators for $C_{0(2)}^{ess}(\theta_0)$ and $C_{0(2)}^{egp}(\theta_0)$ obtained above, using the methodology presented in Section 3.

**Stage 2**
- simulate GARCH-in-mean series of length $n$ by generating iid innovations $\varepsilon_i$ from the mixture distribution given by (21) with parameters from Tables 1 and 2, $i = 1 \ldots n$;
- fit the GARCH model parameters by QMLE yielding \( \hat{\theta}_n \); let \( \hat{\varepsilon} = (\hat{\varepsilon}_i)_{1 \leq i \leq n} \) be the n-dimensional vector of standardized residuals;

- generate \( \varepsilon^\ast_{t+1}, \ldots, \varepsilon^\ast_T \) from a nonparametric kernel density estimator based on the standardized residuals \( \hat{\varepsilon}_i \) with \( i = 1 \ldots n \);

- using standardized versions of \( \varepsilon^\ast_s \) with \( t + 1 \leq s \leq T \) compute the estimators \( \hat{C}_{0(n)}^{\text{ess}}(\hat{\theta}_n) \) and \( \hat{C}_{0(n)}^{\text{egp}}(\hat{\theta}_n) \) using the Monte Carlo simulation.

The Stage 2 algorithm is replicated \( L \) times. The sample mean and the sample variance of the estimators obtained are also reported in the last two columns of Table 3 and Table 4.

We compute prices for options with time to maturity \( T-t=30 \) days and strike prices ranging from deep out-of-money to deep in-the-money\(^*\). The initial price is taken to be \( S_0 = 925 \). The number of pricing estimators from Stage 2 is \( L=100 \). We compute Monte Carlo prices based on \( M=50,000 \) simulated stock paths. As mentioned in the previous section we use a Gaussian kernel with the optimal bandwidth given by (20). The number of normal components for this density is \( n=5,000 \), the number of observed standardized residuals.

Table 3: European Call option prices using the Esscher transform; the true prices and their estimated variances are reported in the second and the third column while the sample mean and the sample variance of the semiparametric estimators are given in columns 3 and 4 ( \( T-t = 30 \), \( S_0 = 925 \), \( M = 50,000 \), \( n = 5000 \), \( L = 100 \))

<table>
<thead>
<tr>
<th>Strike</th>
<th>( C_{0(2)}^{\text{ess}} )</th>
<th>( \hat{v}_{M}^{\text{ess}} )</th>
<th>( C_{0(n)}^{\text{ess}} )</th>
<th>( \hat{v}_{L}^{\text{ess}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>850</td>
<td>81.3528</td>
<td>0.02</td>
<td>81.3962</td>
<td>0.0402</td>
</tr>
<tr>
<td>900</td>
<td>39.8428</td>
<td>0.0181</td>
<td>39.8714</td>
<td>0.05</td>
</tr>
<tr>
<td>925</td>
<td>24.3466</td>
<td>0.013</td>
<td>24.3547</td>
<td>0.0392</td>
</tr>
<tr>
<td>950</td>
<td>13.4039</td>
<td>0.0077</td>
<td>13.4164</td>
<td>0.0417</td>
</tr>
<tr>
<td>1000</td>
<td>3.0296</td>
<td>0.0017</td>
<td>3.1058</td>
<td>0.0123</td>
</tr>
</tbody>
</table>

The results, which are summarized in Tables 3 and 4, suggest that the semiparametric algorithm performs very well, as expected. For example, considering the absolute relative bias of the estimators \( \hat{C}_{0(n)}^{\text{ess}}(\hat{\theta}_n) \) and \( \hat{C}_{0(n)}^{\text{egp}}(\hat{\theta}_n) \), we remark that the largest one is around 2.5% in the

\(^*\)We chose this example only including this short maturity options because the speed of our programs decays considerably as the time to maturity \( T-t \) and the number of replications \( L \) increase; however, from other numerical simulations we believe this thing should not make a big difference
Table 4: European Call option prices using the extended Girsanov principle; the true prices and their estimated variances are reported in the second and the third column while the sample mean and the sample variance of the semiparametric estimators are given in columns 3 and 4 ($T - t = 30$, $S_0 = 925$, $M = 50,000$, $n = 5000$, $L = 100$).

<table>
<thead>
<tr>
<th>Strike</th>
<th>$C_0^{gp}(2)$</th>
<th>$v_M^{gp}$</th>
<th>$\hat{C}_0^{gp(n)}$</th>
<th>$v_L^{gp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>850</td>
<td>81.1647</td>
<td>0.0235</td>
<td>81.1320</td>
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</tr>
<tr>
<td>900</td>
<td>39.6195</td>
<td>0.0187</td>
<td>39.6356</td>
<td>0.0554</td>
</tr>
<tr>
<td>925</td>
<td>24.1423</td>
<td>0.0131</td>
<td>24.1628</td>
<td>0.0659</td>
</tr>
<tr>
<td>950</td>
<td>13.2217</td>
<td>0.0076</td>
<td>13.2461</td>
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</tr>
<tr>
<td>1000</td>
<td>2.8883</td>
<td>0.0016</td>
<td>2.9484</td>
<td>0.0162</td>
</tr>
</tbody>
</table>

Esscher transform case and around 2.1% for the extended Girsanov principle. We also observe that the sample variances of the distribution of the semiparametric estimators is relatively small in all the cases. Thus, we can conclude that our pricing algorithm is demonstrated to be a valid approach, so in Section 6 we test its performance on other option pricing models existing in the literature.

5 Sensitivity of Estimated Option Prices to the Choice of the Kernel and Bandwidth

In Section 4 we considered our nonparametric density estimator with normal kernel and bandwidth given by (19) and (20). A natural question is how the form of the kernel function and the choice of the bandwidth may affect the option valuation. It is well known that the choice of the kernel and bandwidth should not be split into two different problems. However, from the practical point of you, the choice of the bandwidth is a more important issue to deal with then the choice of the kernel.

Properties of the kernel density estimate depends on the smoothness of the underlying true density. A well known result from nonparametric density estimation theory states that for all smooth distributions the Epanechnikov kernel is $L^2$ asymptotically optimal. The Epanechnikov kernel is

$$k(x) = \frac{3}{4}(1 - x^2), \quad -1 \leq x \leq 1.$$  (22)
There are different optimality criteria for choosing a bandwidth for the Epanechnikov kernel. Here we refer to Hall and Wand (2000) to obtain an optimal window by minimizing an $L^1$ and $L^2$ distance between the true and the estimated distribution. They obtain optimal bandwidths of $h_{E}^{L^2} = 2.345\hat{\sigma}n^{-0.2}$ under the $L^2$ criterion, and $h_{E}^{L^1} = 2.279\hat{\sigma}n^{-0.2}$ under the $L^1$ criterion, where $\hat{\sigma}$ is the empirical standard deviation of the variables $\varepsilon_t$, and in the case that the reference or true density is normal. In practice we use the residuals $\hat{\varepsilon}_t$.

In this simulation study we compare three option price estimators, the first one computed using the semiparametric algorithm from Section 3 based on the $L^2$-optimal bandwidth Gaussian kernel and the other two based on the Epanechnikov distribution with the above two bandwidths.

We use the same GARCH-in-mean model specification as in Section 4 with the parameters estimated in Table 1, but now we do not make any specific assumption on the on the parametric form of the innovation distribution. This distribution has to be estimated from the time series of standardized residuals calculated after QMLE fitting. In the S&P data used here $n=4,290$. The last estimate for the conditional volatility is used as the starting value for the variance process and is $\sigma_0 = 4.0554 \times 10^{-5}$. We consider maturities of 30, 90 and 180 trading days. Since the extended Girsanov principle runs much faster than the Esscher transform we do not repeat this experiment for the latter measure. The results are reported in Table 5.

Table 5 suggests that in general the choice of the Kernel and the bandwidth is not crucial in terms of option valuation. The prices calculated using these three methods are not significantly different. Also the pricing errors does not change across maturities and strike prices. The Monte Carlo estimated variances are in the same order for all models. Thus we conclude that the choice of a particular kernel in our pricing methodology is not extremely critical. However, the choice of bandwidth should be reasonable otherwise one may obtain quite different results.

6 Empirical Analysis on Pricing S&P500 Call Options

In this section we investigate the pricing performance of the our semiparametric GARCH model under both risk neutral transformations presented above.

Some natural questions we wish to answer are (i) how the option prices computed using the kernel density estimator method compare with the observed market prices, and (ii) how these compare with prices based on normal innovations. Another aspect of interest is the behavior of the prices regarding which risk neutral measure we use. In Section 3 we have shown that for GARCH models with normal driving noise the Esscher transform and the extended Girs-
Table 5: Semiparametric Monte Carlo option prices and their estimated variances under different kernel functions and bandwidth; the method used is the Extended Girsanov principle ($S_0 = 925, M = 50,000, n=4.290$)

<table>
<thead>
<tr>
<th>T-t</th>
<th>Strike</th>
<th>$C(h_G^{L^2})$</th>
<th>$v(h_G^{L^2})$</th>
<th>$C(h_E^{L^2})$</th>
<th>$v(h_E^{L^2})$</th>
<th>$C(h_E^{L^1})$</th>
<th>$v(h_E^{L^1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
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<td>79.3931</td>
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<td>0.005</td>
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<td>0.007</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>5.6975</td>
<td>0.003</td>
<td>5.8539</td>
<td>0.003</td>
<td>5.6232</td>
<td>0.003</td>
</tr>
<tr>
<td>180</td>
<td>850</td>
<td>107.1771</td>
<td>0.042</td>
<td>107.4142</td>
<td>0.031</td>
<td>107.0322</td>
<td>0.031</td>
</tr>
<tr>
<td></td>
<td>900</td>
<td>68.9280</td>
<td>0.030</td>
<td>69.0023</td>
<td>0.0256</td>
<td>68.705</td>
<td>0.025</td>
</tr>
<tr>
<td></td>
<td>925</td>
<td>52.9093</td>
<td>0.025</td>
<td>52.8965</td>
<td>0.023</td>
<td>52.7042</td>
<td>0.022</td>
</tr>
<tr>
<td></td>
<td>950</td>
<td>39.3659</td>
<td>0.020</td>
<td>39.3274</td>
<td>0.019</td>
<td>39.1845</td>
<td>0.018</td>
</tr>
<tr>
<td></td>
<td>1000</td>
<td>19.8099</td>
<td>0.011</td>
<td>19.9011</td>
<td>0.011</td>
<td>19.7275</td>
<td>0.011</td>
</tr>
</tbody>
</table>

The Girsanov principle coincide, but for other distributions they may lead to different answers. The methodology described in Algorithm 1 is applied to two different option data sets written on S&P 500 stock index. The parameters for the various GARCH models are estimated under the physical measure using the QMLE. The choice of the length of the return data series is of special interest since we are interested in both the accuracy of the GARCH parameter estimates, and in obtaining an accurate kernel density estimator for the driving noise distribution. On the other hand, we need to address to the high computational time problem when we construct the density estimator based on a large data set. All the simulations are done using C++ using a CPU at 2.4Ghz, and the running time for pricing using an Esscher transform ranges from 3 to 24 hours depending on what maturity is used. However, as computing speeds improve or the availability of parallel or cluster computing improves, this will be of less concern.
6.1 Option data set 1

We perform a short study using the European Call option data taken from Schoutens 2003). This data consists of 54 European Call options on this index at the close of the market on April 18, 2002. The closing price on that day was $S_0 = \$1124.47$, the annual risk free rate is $r = 1.9\%$, and the dividend yield is $d = 1.2\%$. The strike prices ranges from $\$ 975$ to $\$ 1325$ and we consider options with maturities $T = 22, 46, 109, 173, \text{and} 234$ days. The average option price is $\$56.94$. The model parameters are estimated using daily closing prices of $S&P 500$ from January 02, 1988 to April 17, 2002, for a total of 3,606 observations. In general, this information may not be sufficient for option valuation. Therefore, many studies showed improvement in the pricing methodology by including also information about historical option prices. In this simulation study we show that in some cases our semiparametric option pricing model performs quite well even though the prices are computed using only historical information about the returns.

We consider the returns are modeled by

$$y_t = r - d + \lambda \sqrt{h_t} + \sqrt{h_t} \epsilon_t.$$  

In the numerical experiments discussed in Sections 4 and 5 we considered only the simple GARCH volatility specification $\omega(\epsilon_t) = \epsilon_t^2$. In a recent study Christoffersen and Jacobs (2004) found that an asymmetric model constructed by including a leverage effect in the variance equation performs the best in terms of pricing European options when the driving noises are normally distributed. Thus, we want to test also the pricing performance when the source of asymmetry in our model comes not only from the returns innovation distribution, but from the volatility equation. Therefore, we consider two GARCH models for volatility: (i) the simple GARCH and (ii) the threshold GARCH (TGARCH) model introduced by Glosten et al (1993)

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 h_{t-1} + \beta_1 h_{t-1}$$

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 h_{t-1} + \gamma I(\epsilon_{t-1} < 0) \epsilon_{t-1}^2 h_{t-1} + \beta_1 h_{t-1}$$

where $I$ is the indicator function.

We compute option prices for eight different models:

- **GARCH** - the simple GARCH model with normal innovations;

- **TGARCH** - the threshold GARCH model with normal innovations;
- **EGP-GARCH** - in this case we evaluate the option prices using the semiparametric technique in Algorithm 1 using the measure $Q^{egp}$ obtained from the extended Girsanov principle;

- **ESS-GARCH** - we apply Algorithm 1 this time using the Esscher transform $Q^{ess};$

- **EGP-TGARCH** - the semi parametric threshold GARCH using the extended Girsanov principle;

- **ESS-TGARCH** - the semi parametric threshold GARCH using the Esscher transform;

- **ShGa-GARCH** - the simple GARCH model of Siu *et al.* (2004) with shifted gamma driving noise

- **ShGa-TGARCH** - the threshold GARCH model of Siu *et al.* (2004) with shifted gamma driving noise

The GARCH and TGARCH model parameter estimates and their standard errors are reported in Table 6.

Table 6: GARCH and TGARCH parameters estimated by QMLE using daily closing prices of S&P 500 from January 02, 1988 to April 17, 2002

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\beta_1$</th>
<th>$\lambda$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH</td>
<td>$5.2 \cdot 10^{-7}$</td>
<td>0.042</td>
<td>0.953</td>
<td>0.061</td>
<td>(10$^{-7}$)</td>
</tr>
<tr>
<td></td>
<td>(3.2 $\cdot 10^{-3}$)</td>
<td>(3.7 $\cdot 10^{-3}$)</td>
<td>(0.016)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TGARCH</td>
<td>$1.65 \cdot 10^{-6}$</td>
<td>7.7 $\cdot 10^{-3}$</td>
<td>0.927</td>
<td>0.045</td>
<td>0.093</td>
</tr>
<tr>
<td></td>
<td>(1.8$\cdot 10^{-7}$)</td>
<td>(6.2 $\cdot 10^{-3}$)</td>
<td>(1.7 $\cdot 10^{-2}$)</td>
<td>(8.6 $\cdot 10^{-3}$)</td>
<td></td>
</tr>
</tbody>
</table>

The annualized unconditional volatility is 0.1599 for the GARCH models and 0.1480 for the TGARCH models, while the estimated volatility on April 17, 2002 is 0.1698 and 0.1712 respectively. These last two numbers are used as a starting values in constructing the variance processes according to each volatility dynamic. Even though the normal GARCH and TGARCH models has closed form under the risk neutral measure induced by an Esscher transform, we compute prices by simulating stock paths under the underlying measure $P$ due to the variance reduction property of this transformation. The option prices for the Shifted Gamma TGARCH model are computing accordingly to Siu *et al.* (2004). Their risk neutralized dynamic is obtained
by applying the Esscher transform. We also adopt their two-stage estimation strategy: the GARCH parameters are obtained by QMLE, while the shape parameter is obtained using the standardized residuals obtained from the QMLE estimation. For our returns data set we find the shape parameter for Gamma density is 22.35 for the simple GARCH volatility dynamic and 30.88 for the TGARCH representation.

The performance of all models is measured by three indicators: (i) the dollar root mean squared error (RMSE), (ii) the average relative pricing error (ARPE) and (iii) the average absolute error (APE) described below.

\[
\text{RMSE (\$)} = \sqrt{\frac{1}{NO} \sum_{j=1}^{NO} (C_{\text{market}}^j - C_{\text{model}}^j)^2}
\]

\[
\text{ARPE (\%)} = \frac{1}{NO} \sum_{j=1}^{NO} \frac{|C_{\text{market}}^j - C_{\text{model}}^j|}{C_{\text{market}}^j} \times 100
\]

\[
\text{APE (\%)} = \frac{1}{NO \cdot C_{\text{market}}} \sum_{j=1}^{NO} |C_{\text{market}}^j - C_{\text{model}}^j| \times 100
\]

where \(NO\) represents the total number of options and \(C_{\text{market}}\) is the average option price.

Table 7 summarizes the overall pricing errors of the various models considered.

<table>
<thead>
<tr>
<th>Model</th>
<th>RMSE</th>
<th>ARPE(%)</th>
<th>APE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH</td>
<td>4.298</td>
<td>8.55</td>
<td>5.96</td>
</tr>
<tr>
<td>TGARCH</td>
<td>3.948</td>
<td>6.80</td>
<td>5.44</td>
</tr>
<tr>
<td>EGP-GARCH</td>
<td>4.754</td>
<td>6.81</td>
<td>6.18</td>
</tr>
<tr>
<td>ESS-GARCH</td>
<td>2.474</td>
<td>7.35</td>
<td>3.73</td>
</tr>
<tr>
<td>EGP-TGARCH</td>
<td>3.218</td>
<td>5.93</td>
<td>4.36</td>
</tr>
<tr>
<td>ESS-TGARCH</td>
<td>1.274</td>
<td>3.34</td>
<td>1.90</td>
</tr>
<tr>
<td>ShGa-GARCH</td>
<td>5.582</td>
<td>8.83</td>
<td>7.31</td>
</tr>
<tr>
<td>ShGa-TGARCH</td>
<td>7.505</td>
<td>11.45</td>
<td>9.83</td>
</tr>
</tbody>
</table>

Upon viewing the results, we remark that, as also indicated by Christoffersen and Jacobs (2004), the TGARCH model outperform the GARCH model for conditionally normal innovations. The same feature also holds for the semiparametric pricing algorithms using either Esscher transform or extended Girsanov principle.

One of our objectives was to study the differences in prices when using the kernel density
estimator for the innovation distribution. The results showed that this may also depend on the choice of the risk neutral measure used. In the simple GARCH model family, the option prices using the Esscher transform, are the closest to the market prices in almost all cases. The improvement of this model over the one with normal innovations is of $1.824 in terms of RMSE and by 38% in terms of APE. However, GARCH slightly outperform the EGP-GARCH in terms of RMSE and APE, while for the threshold model EGP-TGARCH performs better than TGARCH for all three indicators. We notice that ESS-TGARCH is the best option pricing model from the threshold processes. The RMSE is reduced by $2.674 when using the ESS-TGARCH model instead of TGARCH for an average option price of $56.94. Interestingly the highest values in terms of all three indicators are obtained for the Shifted Gamma models. The big differences in RMSE are explained by the poor performance of this models in pricing long maturity options, and this is illustrated in Figure 1 (g) and (h). We can conclude that the conditional Shifted Gamma distribution is not an appropriate choice for modeling the historical return series considered in this example, this making our semiparametric approach even more appealing.

The behavior of the option prices regarding moneyness and maturity is illustrated in Figure 1. EGP-GARCH is the best model to price short maturity options (T=22, 46 days); they also price relatively well deep out-of-money options compared to GARCH or TGARCH. On the other hand, it performs poorly for pricing long maturity options especially for deep in-the-money options, which are severely underpriced as maturity increases. In opposition, ESS-GARCH tends to slightly overprice short maturity deep out-of-money options. Their overall behavior over the GARCH/TGARCH and EGP-GARCH/TGARCH is that they price better the in-the-money and at-the-money options. The results from Table 7 relative to the performance of ESS-TGARCH are strengthened by Figure 1 (f). The most interesting aspect is that this model is able to replicate the market behavior for medium and long maturity options independent of moneyness.

In Figure 2 we presented the pricing errors of all models as functions of strikes. The same conclusions can also be drawn by looking at this pictures. The EGP-GARCH errors are relatively close to zero whenever T is small or $K >> S_0$. The largest error in absolute value is of around $13 and it is obtained when $T = 234$ days and $K << S_0$. The bounds of the pricing errors are around -$6 and $4 for ESS-GARCH and -$1 and $3 for ESS-TGARCH.
Figure 1: European Call option prices evaluated on 18 April 2002 for different models with the maturities are $T = 22, 46, 109, 173,$ and $234$ days; in each panel, the inner series of prices correspond to $T=22$ days the closing stock price on that day was $S_0 = \$1124.47$. 
Figure 2: Model pricing errors for European Call options on 18 April 2002.
6.2 Option data set 2

In this section we test the out-of-sample performance of our pricing methodology. For estimation purposes we consider S&P500 daily index closing prices from January 02, 1988 to June 18, 2003 for a total of 3,900 observations. Our objective is to compute prices for European Calls from June 18, 2003 through June 02, 2004. Our option data consisting of contracts with expiration date on June 19, 2004 was taken from the Bloomberg database. In our numerical comparison we consider options for the last Wednesday of each month during this period (in some cases we took the first Wednesday of the month). We choose this sampling strategy in order to uniformly cover the whole spectrum of maturities in our data set. The maturities range from 17 to 248 days for 12 Wednesdays during the year considered. We consider dividend adjusted stock prices \( e^{-dt}S_t \), where \( d \) is the average dividend yield determined from the forward price. For the risk-free rates we use continuously compounded Treasury Bill rates interpolated to match the maturity of the option. We have also considered only contracts with bid-ask midpoint at the closing trading day greater than the lower bound derived in Merton (1976), and forward moneyness between 0.9 and 1.1. This filtering procedures leave us with 120 contracts with average market price of $ 41.93.

Since from the previous section we noticed that the TGARCH specification for the conditional variance outperforms the simple GARCH one we compute prices only for threshold GARCH models. The parameters are again estimated by QMLE and are reported in Table 8. Given

<table>
<thead>
<tr>
<th>TGARCH</th>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \beta_1 )</th>
<th>( \lambda )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 1.38 \cdot 10^{-6} )</td>
<td>( 8.6 \cdot 10^{-3} )</td>
<td>( 0.931 )</td>
<td>( 0.045 )</td>
<td>( 0.091 )</td>
</tr>
<tr>
<td></td>
<td>( (1.5 \cdot 10^{-7}) )</td>
<td>( (5.9 \cdot 10^{-3}) )</td>
<td>( (5.4 \cdot 10^{-3}) )</td>
<td>( (1.6 \cdot 10^{-2}) )</td>
<td>( (8.4 \cdot 10^{-3}) )</td>
</tr>
</tbody>
</table>

these values and the last estimate for the conditional volatility, we construct time series of volatilities using the index series from the out-of-sample period.

We restrict our attention on comparing our semiparametric option pricing algorithm relative to the normal innovations case (we discarded the Shifted Gamma models due to their poor performance illustrated by the previous example). We take the same dynamic for the returns as in the previous section. The pricing performance is reported in Table 9.
Table 9: Out of sample pricing errors for European Call options June 18, 2003 - June 02, 2004

<table>
<thead>
<tr>
<th>Model</th>
<th>RMSE</th>
<th>ARPE(%)</th>
<th>APE (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TGARCH</td>
<td>3.51</td>
<td>10.02</td>
<td>6.08</td>
</tr>
<tr>
<td>EGP-TGARCH</td>
<td>3.19</td>
<td>8.23</td>
<td>5.32</td>
</tr>
<tr>
<td>ESS-TGARCH</td>
<td>1.65</td>
<td>6.45</td>
<td>2.82</td>
</tr>
</tbody>
</table>

We notice again that the choice of Esscher transform combined with our semiparametric methodology gives the lowest errors for all three indicators used. The dollar root mean squared error for the semiparametric Esscher is 1.65 compared to 3.51 of the normal TGARCH. However, our method just slightly outperform the normal TGARCH when we use the extended Girsanov principle. This behavior is also illustrated in Figure 3 where we plot the Black-Scholes implied volatilities obtained using the observed market prices and Monte-Carlo simulated model prices. The plot suggests that the semiparametric Esscher model implied volatilities capture the volatility smile much better than the other two alternatives. The differences between the implied volatilities for the normal TGARCH and the semiparametric EGP are not so pronounced unless we go deep in-the-money.

7 Conclusions and Future Research

In this paper we propose a new methodology for option pricing in discrete time GARCH models. Our method differs from the other approaches studied in the last several years in the financial literature, in that we do not require any specific parametric distributional assumption for the driving noise. Instead, we approximate the unknown innovation distribution using a kernel density estimator based on the standardized residuals; the advantage of using this nonparametric estimator over a parametric specification for the unknown returns distribution is that the latter may provide a good fit for some stocks and a poor fit for others. In contrast, the estimator based on the residuals cannot fail in these situations. We have also showed that the choice of the kernel and a sensible bandwidth does not influence the results, so one may want to use a kernel function which is easier to implement.

Our empirical studies on S&P 500 European Call option pricing demonstrate the fact that the choice of the risk neutral measure is crucial in option pricing. In our algorithm we have constructed option prices based on two well-known principles: the Esscher transform and extended
Figure 3: Implied volatility smiles based on MLE estimates using returns from January 04, 1988 to June 18, 2003

Girsanov principle. Both measures represent natural choices because they can be justified by some economic arguments. For some cases, these risk neutral measures coincide, for example in the GARCH model with Gaussian innovations. We proved that in this Gaussian noise case Duan’s LRNVR and both Esscher and extended Girsanov principle are the same. The Esscher and extended Girsanov principle also solve a minimal entropy problem when we try to find risk neutral measures within a special class of densities. However, when we no longer assume conditional normality for the returns, the two martingale measures are not identical.

The semiparametric pricing algorithm leads to some very encouraging results. First, we numerically showed that both the Esscher transform and extended Girsanov principle for semipara-
metric leverage GARCH models outperform the usual normal leverage ones. Another appealing conclusion is that our pricing methodology based on the Esscher transform for the threshold GARCH model outperforms by far all the other GARCH models we studied. When using the first set of options we numerically show that the normal GARCH performs better overall than the Shifted Gamma GARCH proposed by Siu et al. (2004). However, the latter provide a better fit for short maturity options. The out-of-sample performance illustrated in our second example also shows that the semiparametric option pricing model gives the closest option prices relative to the observed market ones if we use the Esscher transform. An interesting study would be to compare the in-sample and out-of-sample performance of our semiparametric pricing methodology with parameters calibrated to the market prices versus other various discrete and continuous time stochastic volatility models existing in the literature.

Another noteworthy aspect is that unlike the Esscher transform, the extended Girsanov principle tends to provide the better method to price short maturity options amongst all the GARCH models studied here. It may be also interesting to test the difference between these two risk neutral measures and other possible choices of martingale measures, assuming different GARCH models such as the Jump GARCH model proposed by Duan et al. (2006).

It would also be interesting to consider more efficient methods of estimating the GARCH parameters and their resulting effect on the precision of option price estimation. Here we use QMLE estimates. One could instead use semiparametric GARCH estimators based on the non-parametric kernel density used in our algorithm. Another possibility is to use a QMLE method based on a likelihood other than Gaussian.

References


