McMaster University Math 1A03
Fall 2010 Midterm 1 October 7 2010 Duration: 90 minutes

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Name: SOLUTIONS

Student ID Number: ______________

Instructions

– This test paper is printed on both sides of the page. There are 7 questions on pages 2 through 7; page 8 is blank for rough work. For full credit you must show all your work.

– You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator.

– Only the McMaster standard calculator, the Casio fx , is permitted.

– Answers must be written in pen.

Problem	Points
1 [8]	
2 [8]	
3 [9]	
4 [8]	
5 [9]	
6 [4]	
7 [4]	
Total [50]	

1) [8 marks] No partial credit will be given on this question.

a) State the Riemann sum definition of the definite integral.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x,$$

where $\Delta x = \frac{b-a}{n}$, x_i^* is a sample point in the interval $[x_{i-1}, x_i]$, and $x_i = a + i\Delta x$

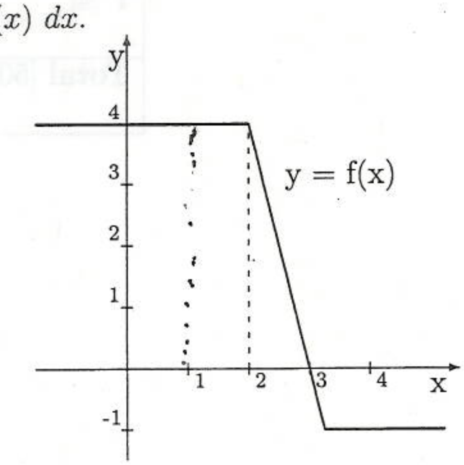
b) $\int_4^9 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\left(4 + i\frac{5}{n}\right)^2$. What is the function $f(x)$?

$f(x) = x^2$

c) State Part I of the Fundamental Theorem of Calculus.

Let $g(x)$ be the function defined on (a,b) by $g(x) = \int_a^x f(t)\,dt$. If f is continuous on $[a,b]$ ~~and~~ then g is differentiable and $\frac{d}{dx} g(x) = f(x)$.

d) For the function whose graph is shown, find $\int_1^3 f(x)\,dx$.



$$\int_1^3 f(x)\,dx = \text{area } \square + \text{area } \triangle$$

$$= 4 \times 1 + \frac{1}{2} \times 1 \times 4$$

$$= 6.$$

2) [8 marks] No partial credit will be given on this question.

a) Find the derivative of $f(x) = x^5 e^{2x}$ (do not simplify your answer).

$$f'(x) = 5x^4 e^{2x} + x^5 2e^{2x}$$

b) Find the derivative of $g(t) = \sin(\sqrt{t})$ (do not simplify your answer).

$$g'(t) = \cos(\sqrt{t}) \cdot \frac{1}{2} t^{-1/2}$$

c) Find the derivative of $f(x) = \frac{\cos(x)}{x^2+1}$ (do not simplify your answer).

$$f'(x) = \frac{(x^2+1)(-\sin(x)) - 2x\cos(x)}{(x^2+1)^2}$$

d) Let $f(x) = g(h(x))$. Suppose that $g(0) = 1, g'(0) = -1, g(5) = 7, g'(5) = 10, h(0) = 5, h'(0) = -2$. Find $f'(0)$.

$$f'(x) = g'(h(x))\, h'(x)$$

$$f'(0) = g'(h(0))\, h'(0)$$

$$= g'(5)(-2)$$

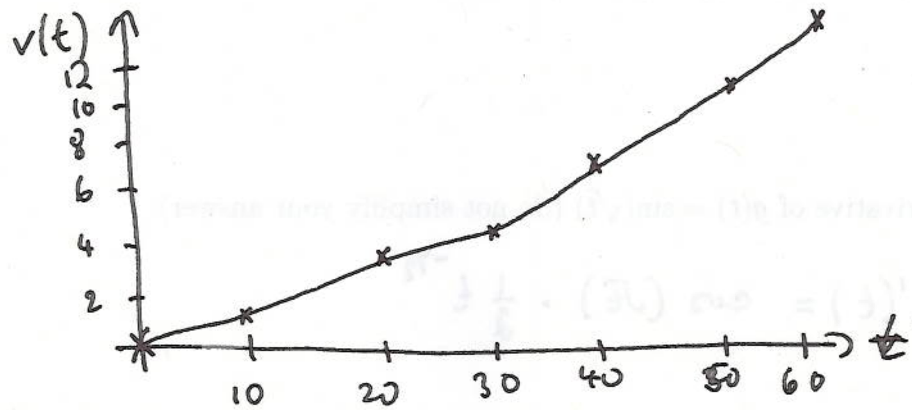
$$= 10 \times (-2)$$

$$= -20$$

3) [9 marks] Consider the following table of velocity/time data over a period of one minute. Assume that the velocity function is increasing over this time.

time in seconds	0	10	20	30	40	50	60
velocity in meters per second	0	1	3	4	6	9	12

a) Sketch a graph of $y = v(t)$.



b) Using left endpoints of ten-second intervals, estimate the distance travelled during this minute. State whether your estimate is too high or too low for the actual distance travelled, and explain your reasoning.

$L_6 = 0\times10 + 1\times10 + 3\times10 + 4\times10 + 6\times10 + 9\times10$

$= 230$

the estimate is too low, because the function is increasing, so the ~~area of the~~ rectangles lie below the graph.

c) Using the given data, calculate a better estimate of the distance travelled. Justify that your method is an improvement.

Calculate R_6, take $\frac{1}{2}(L_6+R_6)$.

R_6 ~~too~~ is too large, so the exact value is somewhere in-between.

Or Use the midpoint rule with 3 intervals. This is a better estimate, as the additional area added in ~~on~~ on the left of each sample point is compensated for by the area left out to the right of the sample point.

4) [8 marks] Find the general form of the following indefinite integrals.

a) $\int e^x \cos(e^x)\, dx$

let $u = e^x$

$du = e^x\, dx$

$$\int e^x \cos(e^x)\,dx = \int \cos(u)\,du$$

$$= \sin(u) + C$$

$$= \sin(e^x) + C$$

b) $\int (x^2-3)\sqrt{x^3-9x}\, dx$

let $u = x^3 - 9x$

$du = (3x^2 - 9)\,dx$

$= 3(x^2-3)\,dx$

$\frac{1}{3}\,du = (x^2-3)\,dx$

$$\int (x^2-3)\sqrt{x^2-9x}\,dx = \int \frac{1}{3}\sqrt{u}\,du$$

$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C$$

$$= 2/9\,(x^3-9x)^{3/2} + C$$

5) [9 marks] Find the following definite integrals.

a) $\int_{-\pi}^{0} \sin(x)\, dx$

$$= \left[-\cos(x)\right]_{-\pi}^{0}$$

$$= -\cos(0) - (-\cos(-\pi))$$

$$= -1 - 1$$

$$= 2$$

b) $\int_{-1}^{1} e^{2x}\, dx$

$$= \left[\frac{1}{2}e^{2x}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right)$$

<u>or</u>

let $u = 2x$
$du = 2\,dx$
when $x = -1$, $u = -2$
$x = 1$, $u = 2$

$$\int_{-1}^{1} e^{2x} dx = \int_{-2}^{2} \frac{1}{2} e^{u}\, du$$

$$= \left[\frac{1}{2}e^{u}\right]_{-2}^{2} = \frac{1}{2}(e^2 - e^{-2})$$

c) $\int_{-3}^{3} x(x^2 - 9)\, dx$

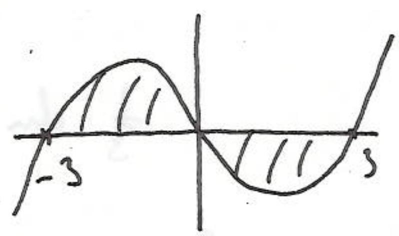
$$= \int_{-3}^{3} (x^3 - 9x)\, dx$$

$$= \left[\frac{1}{4}x^4 - \frac{9}{2}x^2\right]_{-3}^{3}$$

$$= \frac{1}{4}3^4 - \frac{2}{2}3^2 - \left(\frac{1}{4}(-3)^4 - \frac{9}{2}(-3)^2\right)$$

$$= 0$$

<u>or</u>



$$\int_{-3}^{3} x(x^2-9)\,dx = 0$$

6) [4 marks] You decide to plant a bamboo from seed and keep it in your room. Assume that the bamboo grows at a rate of $\frac{3}{2}\sqrt{t}$ meters per year. After how many years will it reach the 8 foot high ceiling?

Let $h(t)$ = height of bamboo after t years.

$h'(t) = \frac{3}{2}\sqrt{t}$.

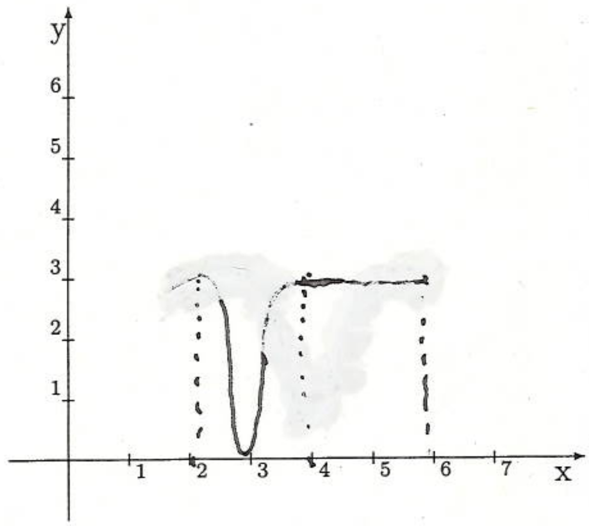
Find T so that $h(T) = 8$. That is, $\int_0^T \frac{3}{2}\sqrt{t}\,dt = 8$

$\left[\frac{3}{2}\cdot\frac{2}{3}t^{3/2}\right]_0^T = 8$

$T^{3/2} - 0^{3/2} = 8$

$T = 4$

7) [4 marks] (a) On the axes given, sketch the graph of a continuous function f with the property that L_2 is a better estimate for $A = \int_2^6 f(x)\,dx$ than L_4, where L_n denotes the approximation of A using n intervals of equal width and left endpoints.



$L_2 = 2\times(f(2)+f(4))$

$L_4 = 1\times(f(2)+f(3)+f(4)+f(5))$

(b) Is it possible to have a function as in (a) which is increasing?

The above example works because $f(3)$ is much smaller than $f(2)$ and $f(4)$, so is a poor estimate for the other values of the function $f(x)$ for $2<x<4$. This is not possible with an increasing function, as $f(2)< f(3) < f(4)$, so $f(3)$ is a better approximation to the values of the function for $2<x<4$, ~~than~~