SOLUTIONS FOR ASSIGNMENT 2 (MATH3X03)

1.4.2. (5pts)

(a) By the definition,

$$|z_1 - z_2|^2 = (\mathrm{Re}z_1 - \mathrm{Re}z_2)^2 + (\mathrm{Im}z_1 - \mathrm{Im}z_2)^2 \geq |\mathrm{Re}z_1 - \mathrm{Re}z_2|^2.$$

It is also true that for any real numbers a, b, $a^2 + b^2 \leq a^2 + b^2 + 2|a||b| = (|a| + |b|)^2$, therefore it follows that

$$(\mathrm{Re}z_1 - \mathrm{Re}z_2)^2 + (\mathrm{Im}z_1 - \mathrm{Im}z_2)^2 \leq (|\mathrm{Re}z_1 - \mathrm{Re}z_2| + |\mathrm{Im}z_1 - \mathrm{Im}z_2|)^2.$$

Taking the square roots of the above two inequalities, we get

$$|\mathrm{Re}z_1 - \mathrm{Re}z_2| \leq |z_1 - z_2| \leq |\mathrm{Re}z_1 - \mathrm{Re}z_2| + |\mathrm{Im}z_1 - \mathrm{Im}z_2|.$$

Note that by the same method, one can also show that

$$|\mathrm{Im}z_1 - \mathrm{Im}z_2| \leq |z_1 - z_2|.$$

(b) If both of the limits on the right exits, say they are u_0 and v_0 respectively, then for all $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$, such that if $|z - z_0| < \delta_1$, then $|u(x, y) - u_0| < \epsilon/2$, and if $|z - z_0| < \delta_2$, then $|v(x, y) - v_0| < \epsilon/2$. Then by Part (a), if $|z - z_0| < \min\{\delta_1, \delta_2\}$, then

$$|f(z) - (u_0 + iv_0)| \leq |u(x, y) - u_0| + |v(x, y) - v_0| < \epsilon.$$

Thus the limit on the left-hand side exists and the equality in question holds. (But it also follows from the fact that the limit of the sum is the sum of the limits given that each limit exists, if you don't want to write too much...)

Conversely, if the limit on the left of the equality exists, say it is f_0, then for all $\epsilon > 0$, there exists some $\delta > 0$ such that if $|z - z_0| < \delta$, then $|f(z) - f_0| < \epsilon$. Then by Part (a) again, when $|z - z_0| < \delta$, we have

$$|u(x, y) - \mathrm{Re}f_0| \leq |f(z) - f_0| < \epsilon,$$
$$|v(x, y) - \mathrm{Im}f_0| \leq |f(z) - f_0| < \epsilon.$$

Thus the limit on the right-hand side exits and the equality in quetion holds as well.

By the above statement, it is clear that $f(z)$ is continuous if and only if $u(x, y)$ and $v(x, y)$ are. In this case the f_0 above is $f(x_0, y_0)$ and $f(x_0, y_0) = u(x_0, y_0) + iv(x_0, y_0)$.

1.4.8. (2pts)

One can see that $f(z) = |z|$ is continuous by Question 1.4.2 (b), since $u(x, y) = \sqrt{x^2 + y^2}$ is continuous and $v(x, y) = 0$ is continuous.

(But one can always use the 'ϵ-δ' language to show it using $||z| - |w|| \leq |z - w|$.)

Date: 27 Jan, 2012. Tatally 40 points.

1.4.14. (3pts)
(a) Open not closed.
(b) Not open but closed.
(c) Neither open nor closed.

1.4.16. (3pts)
(a) Connected not compact.
(b) Connected and compact.
(c) Not connected but compact.

1.5.2. (2pts)
(a) By Proposition 1.5.3 (iv), this function is analytic everywhere, and $f'(z) = 6z + 7$.
(b) By Proposition 1.5.3 (iv), this function is analytic everywhere, and $f'(z) = 8(2z+3)^3$.
(c) By Proposition 1.5.3 (v), this function is analytic except at $z = 3$. By Proposition 1.5.3 (iii), when $z \neq 3$, we have

$$f'(z) = \frac{3(3-z) - (-1)(3z-1)}{(3-z)^2} = \frac{8}{(3-z)^2}.$$

1.5.4. (2pts)
For any fix t_0, We have the following

$$\begin{aligned}\sigma'(t_0) &= \lim_{t\to t_0} \frac{\sigma(t)-\sigma(t_0)}{t-t_0} = \lim_{t\to t_0} \frac{f(\gamma(t))-f(\gamma(t_0))}{t-t_0} \\ &= \lim_{t\to t_0} \frac{f(\gamma(t))-f(\gamma(t_0))}{\gamma(t)-\gamma(t_0)} \frac{\gamma(t)-\gamma(t_0)}{t-t_0} \\ &= \lim_{t\to t_0} \frac{f(\gamma(t))-f(\gamma(t_0))}{\gamma(t)-\gamma(t_0)} \lim_{t\to t_0} \frac{\gamma(t)-\gamma(t_0)}{t-t_0} \\ &= f'(\gamma(t_0))\gamma'(t_0),\end{aligned}$$

which is what is required.

1.5.6. (6pts)
(a) $f'(z_0) = 2$. So f as a conformal map, locally scales by a factor of 2.
(b) $f'(z_0) = 4i^3 + 4 = 4 - 4i = 4\sqrt{2}e^{-\pi/4}$. So f as a conformal map at z_0 rotate locally by $-\pi/4$ and scales by a factor of $4\sqrt{2}$.
(c) $f'(z_0) = -1/(i-1)^2 = -i/2$. So f rotates locally by $-\pi/2$ and scales by a factor of 1/2.

1.5.10. (3pts)
$f(z) = |z| = \sqrt{x^2+y^2} + i0$. It is easy to verify that f does not satisfy the Cauchy-Riemann equations, and the partial derivatives of the real part do not exist at the origin. Thus, f is nowhere analytic.

1.5.16. (3pts)(Not required)

(a) Because f is analytic, it satisfies the Cauchy-Riemann equations. By the equation $au(x,y) + bv(x,y) = c$, we have (by differentiating on both sides)

$$a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x} = 0,$$
$$a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y} = 0,$$

Combining with the Cauchy-Riemann equations, we get

$$a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} = 0,$$
$$a\frac{\partial u}{\partial y} + b\frac{\partial u}{\partial x} = 0,$$

that is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = 0.$$

Since a, b, c are not all 0, at least one of a and b is not zero (otherwise, c would be zero as well). Then

$$\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 \neq 0.$$

That means all the partial derivatives of the the real part and the imaginary part of f are 0. By Proposition 1.5.5, f is a constant on A.

(b) If a, b and c are complex numbers, then one can see that the determinant for the (coefficient-)matrix above is not necessarily non-zero. For example, if $a = 1$ and $b = i$. However, if $a^2 + b^2 = 0$, then $b = \pm ia$. So by the original condition, $au + bv = c$, we have $au \pm iav = c$. Because a, b, c are not all 0, it follows that $a \neq 0$. So we have

$$u \pm iv = \frac{c}{a}.$$

Which says either f or $\bar{f}$ is a constant on A.

We therefore conclude that the result obtained in (a) is still valid if a, b and c are complex constants.

1.6.2. (10pts)

(a) By Proposition 1.6.4, For any choice of branch for the log function, 3^z is entire and has derivative $(\ln 3)3^z$.

(b) This function is obtained by shifting the $\log(z)$ to the left-hand side by 1 unit. So for example, the principle branch has the analytic region $\mathbb{C}\backslash\{x + iy \mid x \leq -1, y = 0\}$. The derivative is $1/(z + 1)$.

(c) By Proposition 1.6.4, for any choice of branch of the log function, $z^{(1+i)} = e^{(1+i)\log(z)}$ is analytic on the domain of the branch of log chosen and the derivative is $(1 + i)z^i$.

(d) $\sqrt{z} = e^{1/2\log(z)}$. So it is analytic on the domain of a (chosen) branch of the log function; the derivative is $1/2z^{-1/2}$.

(e) $\sqrt{z} = e^{1/3\log(z)}$. So it is analytic on the domain of a (chosen) branch of the log function; the derivative is $1/3z^{-2/3}$.

1.6.4. (4pts)

(a) $\lim_{z\to 1} \frac{\log z}{z-1} = (\log z)'\big|_{z=1} = 1/1 = 1$.

(b) Denote by ϵ a real number. Because letting $z = 1+\epsilon$,

$$\lim_{z\to 1} \frac{\bar{z}-1}{z-1} = \lim_{\epsilon\to 0} \frac{1+\epsilon-1}{1+\epsilon-1} = 1,$$

while letting $z = 1+i\epsilon$,

$$\lim_{z\to 1} \frac{\bar{z}-1}{z-1} = \lim_{\epsilon\to 0} \frac{1-i\epsilon-1}{1+i\epsilon-1} = \lim_{\epsilon\to 0} \frac{-i\epsilon}{i\epsilon} = -1,$$

we know that the limit in question do not exist. (Another method is to show that $\bar{z}$ does not satisfy the Cauchy-Riemann equations at $z = 1$.)