Math 3E 06-07 Homework 2 Solutions

2.11 Determine whether $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ given by

$$\left(\frac{a}{b}, \frac{c}{d}\right) \mapsto \frac{a+c}{b+d}$$

is a function.

f is not a function because it is not well-defined.
We can see this in the example:

$$f\left(\frac{1}{2}, \frac{0}{1}\right) = \frac{1+0}{2+1} = \frac{1}{3}$$

$$f\left(\frac{1}{2}, \frac{0}{2}\right) = \frac{1+0}{2+2} = \frac{1}{4}.$$

2.16 (i) $f : X \to Y$, $\{S_i : i \in I\}$ subsets of X.
Show that $f\left(\bigcup_{i \in I} S_i\right) = \bigcup_{i \in I} f(S_i)$.

(ii) $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$. Give example to show could be proper subset.

(iii) f injection. Prove $f(S_1 \cap S_2) = f(S_1) \cap f(S_2)$.

(i) $f\left(\bigcup_{i \in I} S_i\right) = \{y \in Y : \exists x \in \bigcup_{i \in I} S_i \ (y = f(x))\}$ defn of range

$= \{y \in Y : \exists i \in I \ \exists x \in S_i \ (y = f(x))\}$ defn of indexed union

$\bigcup_{i \in I} f(S_i) = \{y \in Y : \exists i \in I \ (y \in f(S_i))\}$

$= \{y \in Y : \exists i \in I \ \exists x \in S_i \ (y = f(x))\}$

Since the order of two existential quantifiers is interchangeable, the two definitions are equivalent.

(ii) Let $y \in f(S_1 \cap S_2)$. Then $y = f(x)$ for some $x \in S_1 \cap S_2$. Then $y \in f(S_1)$ and $y \in f(S_2)$, so $y \in f(S_1) \cap f(S_2)$. Thus $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$

Consider $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$.

Let $S_1 = \{x \in \mathbb{R} : x < 0\} = \mathbb{R}^{<0}$

$S_2 = \{x \in \mathbb{R} : x > 0\} = \mathbb{R}^{>0}$

$S_1 \cap S_2 = \phi$, so $f(S_1 \cap S_2) = \phi$.

But $f(S_1) = \mathbb{R}^{>0}$, $f(S_2) = \mathbb{R}^{>0}$, so

$f(S_1) \cap f(S_2) = \mathbb{R}^{>0}$.

(iii) $f(S_1 \cap S_2) \subseteq f(S_1) \cap f(S_2)$ shown in (ii), so consider $y \in f(S_1) \cap f(S_2)$. Then $y = f(x_1)$ for some $x_1 \in S_1$ and $y = f(x_2)$ for some $x_2 \in S_2$. Since f is injective, $x_1 = x_2$, hence $x_1 \in S_1 \cap S_2$ and $y \in f(S_1 \cap S_2)$. Thus $f(S_1) \cap f(S_2) \subseteq f(S_1 \cap S_2)$, as required.

2.23 $\sigma \in S_n$, $\sigma(j) = j$. Let $X = \{1, 2, \dots n\} \setminus \{j\}$ and define $\sigma' \in S_X$ by $\sigma'(i) = \sigma(i)$ for all $i \in X$. Show $\mathrm{sgn}(\sigma') = \mathrm{sgn}(\sigma)$.

Write $\sigma = \alpha_1 \dots \alpha_r$ as a product of disjoint (non-trivial) cycles. Since $\sigma(j) = j$, j does not appear in $\alpha_1, \dots, \alpha_r$, so each $\alpha_i \in S_X$. Then $\sigma' = \alpha_1 \dots \alpha_r$ as well.

Now $\operatorname{sgn}(\alpha_1 \ldots \alpha_r) = \operatorname{sgn}(\alpha_1) \ldots \operatorname{sgn}(\alpha_r)$ by Prop 2.39.
By the definition of sgn given in class, it does not matter whether α is in S_n or S_{n-1}; the sign is still the same. Thus $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma')$.
Note that the definition given in the text will require a slightly different argument.

2.26 Show that an r-cycle is an even permutation if and only if r is odd.

$$\alpha = (a_1 a_2 \ldots a_r) = (a_1 a_r)(a_1 a_{r-1}) \ldots (a_1 a_3)(a_1 a_2)$$

Thus $\operatorname{sgn}(\alpha) = (-1)^{r-1} = \begin{cases} 1, & r \text{ odd} \\ -1, & r \text{ even} \end{cases}$

Hence α is even iff r is odd.

2.31 (i) $\alpha \in S_n$ Prove $\alpha(i) \neq i$ iff $\alpha^{-1}(i) \neq i$.
(ii) $\alpha, \beta \in S_n$ disjoint. If $\alpha\beta = \varepsilon$ then $\alpha = \varepsilon$ and $\beta = \varepsilon$.

(i) Suppose $\alpha(i) = j \neq i$. Then $\alpha^{-1}(\alpha(i)) = \alpha^{-1}(j) = i$. Since α^{-1} is injective, $\alpha^{-1}(i) \neq i$.
Similarly, suppose $\alpha^{-1}(i) = j \neq i$. Then $\alpha(\alpha^{-1}(i)) = i = \alpha(j)$. As α is injective $\alpha(i) \neq i$.

(ii) If $\alpha\beta = \varepsilon$ then $\beta = \alpha^{-1}$. By (i), α and β move the same elements. Since α, β are disjoint they cannot both move the same element. Hence both α and β move no elements i.e. $\alpha = \beta = \varepsilon$.

2.33 $\alpha = (1\ 2)$ $\beta = (3\ 4)$ $\gamma = (3\ 5)$

$\alpha\beta = \beta\alpha$ as they are disjoint

$\alpha\gamma = \gamma\alpha$ as they are disjoint

$\beta\gamma = (3\ 5\ 4) \neq (3\ 4\ 5) = \gamma\beta$.

2.35 Can ~~the~~ given puzzle be won?

By Example 2-36 and 2.42, the puzzle can be won if and only if the permutation α which represents its starting position is an even permutation which fixes #. Here,

$$\alpha = (1\ 4)(2\ 10\ 5\ 8\ 6)(3\ 9\ 12)(7\ 15\ 13)(11)(14)(16)$$

$\mathrm{sgn}(\alpha) = (-1)(-1)^5(-1)^3(-1)^3 = +1$, and α fixes #, hence the puzzle can be won.

2.37 Prove that $(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1} a_1^{-1}$, in a group.

By definition, $(a_1 a_2 \cdots a_n)^{-1}$ is the element which satisfies $(a_1 a_2 \cdots a_n)^{-1}(a_1 a_2 \cdots a_n) = e$. Check:

$$\begin{aligned}(a_n^{-1} \cdots a_2^{-1} a_1^{-1})(a_1 a_2 \cdots a_n) &= (a_n^{-1} \cdots a_2^{-1})(a_1^{-1} a_1)(a_2 \cdots a_n) \\ &= (a_n^{-1} \cdots a_3^{-1})(a_2^{-1} a_2)(a_3 \cdots a_n) \\ &= e\end{aligned}$$

(by induction, and generalised associativity).

~~**2.38** $\alpha = (1\ 2)(4\ 3)(1\ 3\ 5\ 4\ 2)(1\ 5)(1\ 3)(2\ 3)$~~

~~order(α) = lcm(orders of the cycles) = lcm$(2, 2, 5, 2, 2, 2) = 10$~~

~~(by discussion on Thursday or by Prop 2.55)~~

<u>2·38</u> $\alpha = (1\ 2)(4\ 3)\ (1\ 3\ 5\ 4\ 2)(1\ 5)\ (1\ 3)\ (2\ 3)$

$= (1\ 5\ 4)(2\ 3)$

order of $\alpha = \operatorname{lcm}(3, 2) = 6$ (by discussion on Thursday, or by Prop 2.55)

~~parity of α is~~ $\operatorname{sgn}(\alpha) = (-1)^2 (-1)^1 = -1$, so parity is odd.

$\alpha^{-1} = (2\ 3)^{-1} (1\ 5\ 4)^{-1}$ by 2.37

$= (2\ 3)(1\ 4\ 5)$

(2·22) $\beta = (1\ 9)(2\ 8)(3\ 7)(4\ 6)\ (5)$

order of $\beta = 2$.

(2·28) $f = (0)(1)(2\ 6\ 10\ 7)(3\ 9\ 4\ 5)$

order of $f = 4$.

2.20 The following argument has an error; find it.

"Let R be a relation which is symmetric and transitive on the set X. Let $x \in X$ and suppose xRy. Then by symmetry yRx, so by transitivity xRx. Thus R is reflexive."

The definition of reflexive is <u>for all</u> $x \in X$, xRx.
In order for the above argument to start, it assumes that for the given x, there is a y such that xRy. If there is no such y, then cannot conclude that xRx. Thus the argument proves that, if R is symmetric, transitive and satisfies for all $x \in X$ there is $y \in X$ such that xRy, then R is reflexive.

<u>Ex</u> xRy iff $0 < x < 1$ & $0 < y < 1$.

Then $\neg\, 0Ry$ for any y

$\neg\, 1Ry$ for any y.

For all other x, y xRy trivially. In particular, R is symmetric and transitive.

2.29

(i) Let $\alpha \in S_n$ and suppose $\alpha = (a_1\, a_2 \cdots a_r)$. For any $i \in \{1, \ldots, n\}$, and any k, $\alpha^k(a_i) = a_{i+k \,(\text{mod } n)}$. Thus, if we start to write α^k in cycle notation, we will get $(a_i\ a_{i+k} \cdots a_{i+(r-1)k})$, where r is the least number satisfying $i + rk \equiv i \mod n$

i.e. $n \mid rk$.

i.e. $r = \frac{n}{\gcd(n,k)}$.

In particular, this computation is independent of i, so all cycles are the same length, which is $\frac{n}{\gcd(n,k)}$.

Thus any power of an n-cycle is regular.

Conversely, suppose $\alpha = (a_1^1 \cdots a_r^1)(a_1^2 \cdots a_r^2) \cdots (a_1^\ell \cdots a_r^\ell)$ is regular, so in particular $r\ell = n$. Let

$\beta = (a_1^1\ a_1^2 \cdots a_1^\ell\ a_2^1\ a_2^2 \cdots a_2^\ell\ a_3^1 \cdots\ a_r^\ell)$.

By the above calculation, $\beta^\ell = \alpha$, and hence α is a power of an n-cycle.

(ii) The computation above shows that α^k is a product of ℓ r-cycles, where $r = \frac{n}{\gcd(n,k)}$ and hence $\ell = \gcd(n,k)$.

(iii) Suppose $n = p$ is prime, ~~then~~ By (ii) α^k is a product of $\gcd(p,k)$ - many cycles. $\gcd(p,k) = \begin{cases} 1, & k < p \\ p, & k = p \end{cases}$

thus α^k is one p-cycle if $1 \le k < p$ and p 1-cycles (i.e. the identity) if $k = p$.

(iv) # regular permutations = # distinct n-cycles and their powers. + 1 for the identity

S_5: since 5 is prime, the powers of a 5-cycle are ~~the~~ 5-cycles. Thus it suffices to count the number of distinct 5-cycles. Assume the cycle starts with 1; then each subsequent term can be chosen in 4 (respectively 3, 2, 1) ways.

Hence # regular permutations $= 1 + 4! = 25$.

S_8: If α is an 8-cycle, then $\alpha^3, \alpha^5, \alpha^7$ will also be 8-cycles, and $\alpha^2, \alpha^4, \alpha^6$.

distinct 8-cycles $= 7!$

families of regular permutations $\{\alpha, \alpha^3, \alpha^5, \alpha^7\} = \frac{7!}{4}$

regular permutations of form $\alpha^2, \alpha^4, \alpha^6$ is $\frac{7!}{4} \times 3$

regular permutations $= 7! + \frac{7!}{4} \times 3 + 1$